ISBN-82-553-0614-5 No.4
October 1 1986

SOME EFFICIENT DETERMINISTIC WEAPONS AGAINST COMPLEXITY IN RELIABILITY THEORY: COHERENT SUBSYSTEMS AND PSEUDO-MODULES FOR COHERENT SYSTEMS

***

Part I: The Binary Case

***

by **N. MAZARS**

Work supported by the Royal Norwegian Council for Industrial and Scientific Research (NTNF)

ABSTRACT: In **reliability** theory, modules [3], only, have been proposed for studying complex coherent systems which however may include none of them, except the "trivial" ones. This report is the first part of a study concerned with module generalizations, includes two chapters and is restricted to the binary case.

Chapter A: After a short review of the three basic deterministic treatments for binary coherent systems, the most general notion which can be reasonnably considered for generalizing modules is first introduced with the coherent subsystems. Then, the pseudo-modules are introduced as some special coherent subsystems, by investigating the conditions which are necessary and sufficient for extending the key result concerning modules and their minimal path (cut) sets [3]. Among various fundamental properties, it is shown that indeed, a pseudo-module can always be deduced from a coherent subsystem.

Simple coherent subsystems and pseudo-modules are characterized in various ways, thus extending the well-known characterizations for modules [3][7], such as the "network representations" and the "tests for modularity".

In addition, various special cases for pseudo-modularity are examined: the most efficient pseudo-modular decompositions for complex systems are considered with the strong pseudo-modularity, the strongest case for pseudo-modularity is introduced as the quasi-modularity and the key result concerning **reliability** calculation in terms of modules is extended in terms of monotone pseudo-modules.

Chapter B: After a short review of the "basic bounds" for the interval (un-)reliability, all the "refined bounds" currently proposed in terms of modular decompositions for the interval (un-)reliability and for the (un-)availability [20] are generalized in terms of pseudo-modular decompositions.

All the results proposed in this study concern the binary coherent systems, only, but they should be equally retained as some fundamental results for investigating coherent subsystems and pseudo-modules for multinary coherent systems, with some simple approach. This is shown in the second part of this study, thus extending all the "refined bounds" proposed in [15], with some easier proofs.

Indeed, any coherent system includes some non-trivial pseudo-modules and it can be expected that the notions proposed in this study can appear helpful in some other contexts of systems theory.

KEY WORDS: COHERENT SYSTEMS; MODULES; COHERENT SUBSYSTEMS; SIMPLE COHERENT SUBSYSTEMS; PSEUDO-MODULES; QUASI-MODULES; MONOTONE PSEUDO-MODULES; MUTUAL INDEPENDENCE; ASSOCIATION; INTERVAL RELIABILITY; AVAILABILITY; "REFINED BOUNDS".

CONTENTS

* *
*

Chapter A: New Lights on Modules Through their Generalizations: Coherent Subsystems and Pseudo-Modules for Binary Coherent Systems

* *
*

Chapter B: Pseudo-Modular Decompositions and "Refined Bounds" for the Interval Reliability and for the Availability for Binary Coherent Systems

* *
*

Preamble:

PSEUDO-MODULES GENESIS

> "But often a little clear thought can replace a great deal of calculation."
>
> T. POSTON "Purity in Applications" in "Mathematics Tomorrow" (1981) Edited by L.A. Steen, Springer Verlag, New York, pp.49-54.

Coherent systems [12][4] play a central part in **reliability** theory [1][2] for various reasons [18]. However, the calculation of the **reliability** characteristics for some coherent systems is an NP-difficult problem [21]. Since the well-known paper proposed in 1965 by Birnbaum and Esary [3], modules, only, have been considered for decomposing complex coherent systems. However, they cannot be easily detected from real systems [8][9] and above all, various complex coherent systems do not admit any module, except the "trivial" ones [3].

This report is the first part of a study concerned with module generalizations. The most general notion which can be reasonnably considered for this purpose is first introduced with the coherent subsystems. Among the coherent decompositions then studied, the pseudo-modular decompositions are much more general than the modular decompositions, concern any coherent system but their properties can appear sufficient for extending all the essential results currently proposed for modules [3][7], including the application domain of all the "refined bounds" for the interval reliability and for the (instantaneous) availability obtained till now in terms of modular decompositions [5][20][6][15].

The first part of this study is only concerned with the binary coherent systems. However, coherent subsystems and pseudo-modules can be generalized for the multinary coherent systems in a straight

way and the results proposed for the binary case must also be retained as some fundamental results for generalizing modules in the context of the multinary coherent systems, with some simple approach proposed in the second part of this study.

Indeed, this study is the natural consequence of some unifying viewpoint on multinary coherent systems [18]. This viewpoint has been obtained by investigating further some helpful "bridge" previously proposed between the binary case and the multinary case [4]: any multinary (broad-sense) coherent system can be characterized with some collection of binary coherent systems submitted to some "monotone contraints". This "binary decomposition" [18] allows to establish a one-to-one correspondence between the basic notions defined for the multinary coherent systems and those previously considered for the binary coherent systems, except one [18]: the important concept of module.
Indeed, the notion of module, first introduced for the binary coherent systems [3], can be easily generalized for the multinary case by extending its definition in terms of structure functions to some larger domain and range [6][15]; in addition, all the "refined bounds" currently obtained in terms of modular decompositions for the binary case [20] have been lately generalized for the interval reliabilities and the availabilities for multinary coherent systems [6][15]; the results then obtained and the approach with which they have been put in evidence are somewhat analogous to the ones previously proposed for the binary case. However, a module for some multinary coherent system does not necessarily yield a module for the binary coherent systems which define its binary decomposition [18]. Once these facts noted, it can be expected that there exist some decompositions which are more general than the modular decompositions and the properties of which can appear sufficient for extending all the "refined bounds" currently proposed. This remark is at the origin of this study: the pseudo-modules have been first introduced as some efficient deterministic weapons for extending and unifying the application domain of the "refined bounds" proposed till now in terms of modular decompositions for the interval relia-

bility and the availability for binary and multinary coherent systems [20][15].
This first attempt has been completed by checking that the pseudo-modular decompositions are the most general coherent decompositions in terms of which the application domain of the "refined bounds" currently proposed in terms of modular decompositions can be extended.
The main pragmatical consequences of the results proposed in this study are given in the first chapter as some conclusions. However, they should be investigated further by considering coherent subsystems and pseudo-modules in the context of fault tree or graph analysis: some efficient algorithms should be elaborated for determining the most significant pseudo-modular decompositions which can be detected from a complex (binary or multinary) coherent system.

If necessary, the first part of this study is divided in two chapters and as an indication, theorem 10 in the third section of chapter A is designated as "theorem A.3.10"; any reference to some definition, some proposition, ... is done in the same way; however, the concerned chapter is not mentioned when a reference and its object are stated in the same chapter.

This work has been supported by the Royal Norwegian Council for Industrial and Scientific Research (NTNF).
I wish to thank Pr B. Natvig for having made me aware of the study lately proposed by A. Huseby [17] and for having detected some excessive generalization considered for [20; Lemma A.1.1] in an earlier version of this study.
In addition, it must be mentioned that the existence of the pseudo-modules has been first apprehended in the course of my previous study [18], performed with the support of the Joint Research Centre of the European Communities Commission, at the "System Engineering and Reliability Division" of Ispra Establishment (Italy).

OSLO, April 1986

Thoroughly revised, September 1986

Chapter A:

NEW LIGHTS ON MODULES THROUGH THEIR GENERALIZATIONS: COHERENT SUBSYSTEMS AND PSEUDO-MODULES FOR BINARY COHERENT SYSTEMS

1. INTRODUCTION AND SUMMARY:

Since the well-known paper by Esary and Proschan [12], the binary coherent systems have been studied intensively with various purposes (e.g. see [2]). However, their application domain is limited by three factors of complexity [18]: a "large" number of components, a "non-classical" structure (i.e. a structure which cannot be decomposed easily in some series-parallel structures) and some "strong" stochastic dependences. The two first factors, only, can lead to some NP-difficult problems [21].

Since the well-known paper by Birnbaum and Esary [3], modules, only, have been considered for decomposing complex binary coherent systems. They have been first proposed for simplifying "the minimal path (cut) set method" [12][2] under the mutual independence: if it admits some modular decomposition, a binary coherent system can be studied by considering some smaller order systems themselves coherent. Subsequently, modular decompositions have been proved to be equally helpful for refining [5] [20] various "basic bounds" for the (un-)availability [2] and for the interval (un-)reliability [13][20], under some conditions of dependence including the mutual independence.

However, modules cannot be easily detected from real systems (e.g. see [8][9]). In addition, various systems do not admit any module, except the "trivial" ones. For instance, this is true for the well-known k-out-of-n systems, for the bridge-system of order 5 [2] and more generally, for various coherent systems the minimal path (cut) sets of which include "a lot of repeated components".

In this chapter, generalizations for modules are investigated. In particular, this leads to some collections of subsystems which allow to decompose any coherent system, which are much more general than modules but the properties of which can appear sufficient for extending all the essential results currently proposed for modules [3][7], including the application domain of all the "refined bounds" obtained in terms of modular decompositions [5][20]. Consequently, these generalizations and the related decompositions are introduced as the pseudo-modules and the pseudo-modular decompositions. Their deterministic properties, mainly, are examined throughout this first chapter while the corresponding "refined bounds" are given in chapter B.

First, the binary coherent systems are briefly reviewed in section 2, by adding some nuances to some definitions used most often.

In section 3, generalizations for modules are investigated in terms of structure functions: the most general notion which can be reasonnably considered for this purpose is first proposed with the coherent subsystems; then, the simple coherent subsystems can be introduced by adding some slight refinement and the pseudo-modules can be put in evidence by investigating the conditions which are necessary and sufficient for extending the fundamental result concerning modules: if it admits a modular decomposition, the minimal path (cut) sets of a binary coherent system can be determined from those of its modules and of the corresponding organizing system. In addition, various fundamental properties are proposed and illustrated with some simple systems which are well known for not admitting any module, except the trivial ones: in particular, it is shown that a pseudo-module can always be deduced from a coherent subsystem.

Throughout sections 4 and 5, the simple coherent subsystems and the pseudo-modules are characterized in various ways, thus extending the well-known characterizations for modules [3][7]. In section 4, the generalized "network representations" [3] enlight two interesting features of the pseudo-modules: first, any relevant subset of components can be a pseudo-modular subset; second, as soon as a pseudo-module is identified, a pseudo-modular decomposition is determined. In section 5, some set-characterizations of the simple coherent subsystems and of the pseudo-modules allow to put in evidence the strong pseudo-modules which can yield the most efficient decompositions. In addition, the strongest case for pseudo-modularity is introduced as the quasi-modularity.

In section 6, the monotone pseudo-modules are introduced and it is shown that under the mutual independence, the monotone pseudo-modular decompositions appear helpful for determining the reliability of the complete system, thus extending some well-known result for modules [3].

In section 7, the main pragmatical consequences of the results proposed in this study are given as some conclusions.

Indeed, some enthusiastic introductive arguments proposed in [3] for modules can be (and should be) retained for the generalizations proposed in this study: "the definitions we employ are motivated by the need of reliability analysis" but "we can hope and do suspect" that the simple coherent subsystems and the pseudo-modules can yield some efficient weapons against Complexity "in other contexts of systems theory". As an indication, a binary coherent system is mathematically equivalent to a "simple game" as also to a "blocking system" [7]. In addition, the results proposed in this chapter can be easily extended for the multinary coherent systems: this is shown in the second part of this study [19] (see Preamble).

2. BRIEFLY, THREE BASIC DETERMINISTIC TREATMENTS FOR BINARY COHERENT SYSTEMS:

First introduced in terms of boolean functions [12][2], the binary coherent systems can also be formulated in terms of some continuous real-valued functions [11][2] as also in a set-theoretic context [7]. For clarity sake, these three basic deterministic treatments, unified by the minimal path (cut) set concept, are briefly reviewed by adding some nuances [18] to some definitions used most often: in particular, two types of coherence are distinguished and the relevance concept is introduced in set terms. This extended definition allows to consider modules and their generalizations in the more general context of the broad-sense coherence.

Notations 2.1: Any system to be considered is assumed defined in such a way that its performance level (i.e. 0 for failure, 1 for functioning) can be fully determined from those of its components; so, it can be designated in an abbreviated way as some couple, (C,Φ), where C denotes its component set, assumed finite, and Φ its structure function. For instance, let $C \subset \mathbf{N}^*$.
Let $|A|$ and $\bar{A}$ denote respectively the cardinality of an arbitrary set A and its complement with respect to some reference-set (unambiguous from the context). Let $S = \{0,1\}$. For every subset M of C, let $S_M = S^{|M|}$, $\underline{x}_M = (x_i)_{i \in M} \in S_M$ and $\underline{x} = (\underline{u}_M, \underline{x}_{\bar{M}})$ for some $u \in S$ if and only if $x_i = u$ for every $i \in M$; in particular, $S_\emptyset = \emptyset$, $\underline{x} = (u_i, \underline{x})$ if $M = \{i\}$ and $\underline{x} = \underline{u}$ if $M = C$.
More generally, any vector $\underline{x} \in S_C$ can be specified as follows:
$\underline{x} = (\underline{x}_{M_r})_r \in \overset{m}{\underset{r=1}{\times}} S_{M_r}$ for some partition $\{M_r \;/\; r=1,\ldots,m\}$ of C.
Φ is a function from S_C into S and the usual partial order is considered on S_C: for every $\underline{x} \in S_C$ and $\underline{y} \in S_C$, $\underline{x} \le \underline{y}$ ($\underline{x} < \underline{y}$) if and only if for every $i \in C$, $x_i \le y_i$ (and $x_i < y_i$ for some $i \in C$).

Definition 2.2: A binary system, (C,Φ), is **semi-coherent** if and only if its structure function Φ is non-decreasing (in each of its arguments) from S_C into S.

Definition 2.3: A binary semi-coherent system, (C,Φ), is **broad-sense coherent** if and only if in addition, $\Phi(\underline{0}) = 0$ and $\Phi(\underline{1}) = 1$.

Definition 2.4: A non-empty subset M of C is said to be **relevant** to (C,Φ) if and only if there exists some $\underline{x}_{\bar{M}} \in S_{\bar{M}}$ such that $\Phi(\underline{1}_M,\underline{x}_{\bar{M}}) = 1$ and $\Phi(\underline{0}_M,\underline{x}_{\bar{M}}) = 0$. In the particular case $M = \{i\}$, the component i is said to be relevant to (C,Φ).
Otherwise, M or i is **irrelevant** to (C,Φ).

Definition 2.5: A binary semi-coherent system is **strict-sense coherent** if and only if all its components are relevant.

Remark 2.6: Indeed, two functions of structure only are possible for the binary semi-coherent systems which are not broad-sense coherent: $\Phi \equiv 0$ and $\Phi \equiv 1$. In addition, a non-empty subset of components is relevant to (C,Φ) if and only if it contains at least one component which is itself relevant to (C,Φ) and the strict-sense coherent systems class is strictly included in that of the broad-sense coherent systems.

Convention 2.7: Throughout the following, any so-called binary coherent system is implicitely assumed broad-sense coherent whereas M denotes a non-empty subset of C relevant to (C,Φ).

Definition 2.8: M is a **modular subset** of (C,Φ) if and only if there exist two binary coherent systems, (M,μ) and $(C'=\{i_M\}\cup\bar{M},\varkappa)$, such that for every $\underline{x}\in S_C$, $\Phi(\underline{x}) = \varkappa(\mu(\underline{x}_M)_{i_M},\underline{x}_{\bar{M}})$, for some $i_M \in \mathbf{N}^*$, $i_M \notin \bar{M}$.
Then, (M,μ) is the corresponding **module** of (C,Φ).

The broad-sense coherence property appears immediately sufficient for ensuring the existence of the minimal path (cut) sets:

Definition 2.9: A non-empty subset H (K) of C is a **path (cut) set** of (C,Φ) if and only if $\Phi(\underline{1}_H,\underline{0}_{\bar{H}}) = 1$ $(\Phi(\underline{0}_K,\underline{1}_{\bar{K}}) = 0)$.
In addition, a **minimal path (cut) set** is any path (cut) set which is minimal according to the inclusion relation: for every subset A of C, if $A \subset H$ $(A \subset K)$ then, $\Phi(\underline{1}_A,\underline{0}_{\bar{A}}) = 0$ $(\Phi(\underline{0}_A,\underline{1}_{\bar{A}}) = 1)$.

Notations 2.10: Let $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) be the collection of the minimal path (cut) sets of an arbitrary coherent system (C,Φ). In addition, for every finite non-empty subset I of N^* and every $\underline{y}\in S_I$, let

$$\coprod_{i\in I} y_i = \mathrm{Max}\{y_i \ / \ i\in I\} = y_1 \vee \dots \vee y_{|I|};$$
$$\prod_{i\in I} y_i = \min\{y_i \ / \ i\in I\} = y_1 \dots\dots y_{|I|}.$$

Remark 2.11: The following relations are well known:

(2.1) $\Phi \equiv \coprod_{r=1}^{h} \eta_r \equiv \prod_{r=1}^{k} \theta_r \iff \forall \underline{x}\in S_C,\ \Phi(\underline{x}) = \coprod_{r=1}^{h} \prod_{i\in H_r} x_i = \prod_{r=1}^{k} \coprod_{i\in K_r} x_i$

where $\mathcal{H}(\Phi) = \{H_r \ / \ r=1,\dots,h\}$, $\mathcal{K}(\Phi) = \{K_r \ / \ r=1,\dots,k\}$ for some $(h,k)\in\mathbf{N}^{*2}$ while (H_r,η_r) ((K_r,θ_r)) denotes the series- (parallel-) system corresponding to the minimal path (cut) set H_r, $r = 1,\dots,h$ (K_r, $r = 1,\dots,k$) respectively.

(2.2) By definition of the dual system, (C,Φ^D), for every $\underline{x}\in S_C$, $\Phi^D(\underline{x}) = 1 - \Phi(\underline{1}-\underline{x})$; so, $\mathcal{H}(\Phi^D) = \mathcal{K}(\Phi)$ and $\mathcal{K}(\Phi^D) = \mathcal{H}(\Phi)$.

The minimal path (cut) set concept leads to various set-characterizations for coherence [7]; in particular, the following result plays a fundamental part.

Theorem 2.12: Given a finite set C and a collection $\mathcal{H}$ ($\mathcal{K}$) of non-empty subsets of C, there exists some binary (broad-sense) coherent system, (C,Φ), which admits $\mathcal{H}$ ($\mathcal{K}$) as the collection of its minimal path (cut) sets if and only if:

(2.3) for every $(A,A')\in\mathcal{H}^2$ ($\mathcal{K}^2$), A is not strictly included in A'.

Moreover, (C,Φ) can be uniquely defined from the collection $\mathcal{H}$ ($\mathcal{K}$) as follows: for every $A\subset C$, $\Phi(\underline{1}_A,\underline{0}_{\bar{A}}) = 1$ ($\Phi(\underline{0}_A,\underline{1}_{\bar{A}}) = 0$) if and only if $H\subset A$ ($K\subset A$) for some $H\in\mathcal{H}$ ($K\in\mathcal{K}$).

Notation 2.13: Given an arbitrary subset B of C, let $\mathcal{J}_B$ denote the function defined as follows: for every collection of non-empty subsets of C, $\mathcal{A}$, $\mathcal{J}_B(\mathcal{A}) = \{A\cap B \ / \ A\in\mathcal{A}\}$.

Remark 2.14: In set terms,

(i) a non-empty subset M of C is relevant to (C,Φ) if and only if $H\cap M \neq \emptyset$ ($K\cap M \neq \emptyset$), for some $H\in\mathcal{H}(\Phi)$ ($K\in\mathcal{K}(\Phi)$);

(ii) a broad-sense coherent system, (C,Φ), is strict-sense coherent if and only if the cover of $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) is equal to C;

(iii) M is a modular subset of (C,Φ) if and only if the collections $\mathfrak{I}_M(\mathcal{H}(\Phi))$ and $\mathfrak{I}_M(\mathcal{K}(\Phi))$ gather together the minimal path sets and the minimal cut sets, respectively, of some binary coherent system, (M,μ) [7].

Remark 2.15: When a dynamical viewpoint is considered, any vector $\underline{x}$, $\underline{x}\in S_C$, designates some "value observed" at some fixed point in time from some realization of the components behaviour, $\{\underline{x}(u) \,/\, u\in\bar{\mathbf{R}}^+\}$, where $\bar{\mathbf{R}}^+$ denotes the complete positive "real half-line". Any possible realization of the components behaviour, $\{\underline{x}(u)\in S_C \,/\, u\in\bar{\mathbf{R}}^+\}$, is assumed right-continuous on $\bar{\mathbf{R}}^+$.

Then, the life lengths of the components can be defined as follows: for every $i\in C$, $t_i = \mathrm{Sup}\{u\in\bar{\mathbf{R}}^+ \,/\, x_i(u)=1\}$; so, for every $u\in\bar{\mathbf{R}}^+$, $x_i(u) = J_u(t_i)$ where J_u denotes the indicator-function of the upper interval $(u,+\infty]$ of $\bar{\mathbf{R}}^+$. The life (length) function of (C,Φ) must permit to express its life length from those of its components [11].

Definition 2.16: A function τ from $\bar{R}^{+|C|}$ into $\bar{R}^+$ is the **life function** of some binary coherent system, (C,Φ), if and only if: for every $u\in\bar{R}^+$, $\Phi\circ\underline{J}_u = J_u\circ\tau$, where for every $\underline{t} = (t_i)_{i\in C}\in\bar{R}^{+|C|}$, $\underline{J}_u(\underline{t}) = (J_u(t_i))_{i\in C}\in S_C$.

Various characterizations for coherence have been proposed in terms of life functions [11]. Among them, the next result plays a central part.

Theorem 2.17: Given a finite set C, a function τ from $\bar{R}^{+|C|}$ into $\bar{R}^+$ is the life function of some binary coherent system, (C,Φ), if and only if there exists some collection of subsets of C, $\mathcal{H}$ ($\mathcal{K}$), which satisfies the incomparableness conditions (2.3) and which verifies the following relation :

(2.4) for every $\underline{t}\in\bar{R}^{+|C|}$, $\tau(\underline{t}) = \mathrm{Max}\{\min\{t_i/i\in H\}/H\in\mathcal{H}\}$
$(\tau(\underline{t}) = \min\{\mathrm{Max}\{t_i/i\in K\}/K\in\mathcal{K}\})$.

Remark 2.18: Indeed, this characterization consists in extending to some larger domain and range the general expression of the structure functions for binary coherent systems (see (2.1)). Most of the basic notions related to coherence, including modules, can be characterized in the same way in terms of life functions [12][18].

3. FROM COHERENT SUBSYSTEMS TO PSEUDO-MODULES:

3.1. Minimal Path Sets and Cut Sets Through the Simple Coherent Subsystems:

Pseudo-modules and pseudo-modular decompositions are introduced from the most general notion which can be reasonably considered for generalizing modules. In particular, the next definition is identical to the one proposed in [17] for introducing "modularization".

Convention 3.1: Throughout the following, (C,Φ) denotes some binary (broad-sense) coherent system (see Notations 2.1 and Conventions 2.7). In addition, let $d\in N^*$ and $(d_r)_{r=1,\ldots,m}\in N^{*m}$, for some $m\in N^*$.

Definition 3.2: A **coherent subsystem of degree d** of (C,Φ) is a collection of d (distinct) binary coherent systems defined from M, $\{(M,\mu_r) \;/\; r=1,\ldots d\}$, which satisfies the following conditions:

(i) there exists some binary coherent system, $(C'=D\cup\bar{M},\kappa)$ where $D = \{i_r\in N^* \;/\; i_r\notin\bar{M};r=1,\ldots d\}$, such that:

(3.1) $\forall\, \underline{x}\in S_C,\; \Phi(\underline{x}) = \kappa((\mu_r(\underline{x}_M)_{i_r})_{r=1,\ldots,d},\underline{x}_{\bar{M}})$,

(ii) each component $r\in D$ is relevant to (C',κ).

Then, (C',κ) is the corresponding **organizing system** and M is a **coherent subset of degree** d of (C,Φ).

In addition, a **strict coherent subsystem** is defined with strict-sense coherent systems, only (without any further requirement on the corresponding organizing system).

Definition 3.3: A **coherent decomposition of degree $(d_r)_{r=1,\ldots m}$** of (C,Φ) is a collection of coherent subsystems of (C,Φ), $\{\{(M_r,\mu_{rv_r}) \;/\; v_r=1,\ldots,d_r\} \;/\; r=1,\ldots m\}$, which verifies both of the following conditions:

(i) the collection $\{M_r \;/\; r=1,\ldots,m\}$ is a partition of some subset of C including all the components relevant to (C,Φ);

(ii) for every $\underline{x}\in S_C$, $\Phi(\underline{x}) = \kappa((\mu_{rv_r}(\underline{x}_{M_r}))_{r=1,\ldots,m;v_r=1,\ldots,d_r})$, for some binary coherent system $(C'=\bigcup_{r=1}^{m}\{r\}\times\{1,\ldots,d_r\},\kappa)$, the

organizing system of the concerned coherent decomposition. Furthermore, a **strict coherent decomposition** of (C,Φ) is defined with some strict coherent subsystems of (C,Φ) only and its organizing system is itself strict-sense coherent.

Definition 3.4: A **simple coherent subsystem of degree d** of (C,Φ) **with respect to the collection of its minimal path (cut) sets** is a coherent subsystem of degree d of (C,Φ), $\{(M,\mu_r) \;/\; r=1,\dots,d\}$, such that each minimal path (cut) set of the corresponding organizing system, $(C'=\{i_r \in \mathbf{N}^* \;/\; i_r \notin \bar{M};\ r=1,\dots,d\} \cup \bar{M}, \kappa)$, contains at most a single element of $C' \setminus \bar{M} = \{i_r \in \mathbf{N}^* \;/\; i_r \notin \bar{M};\ r=1,\dots,d\}$.
Then, M is a **simple coherent subset of degree d** of (C,Φ) with respect to the collection of its minimal path (cut) sets.

An informal definition for the simple coherent subsystems can be easily stated by adding some refinements to the intuitive interpretation proposed in [3] for introducing modules: loosely speaking, a simple coherent subsystem of some binary coherent system with respect to the collection of its minimal path (cut) sets is defined with any subset of components which can be organized in some subsystems of their own and which affect the system only through the performance of some of these subsystems, which subsystems depend on the other components then functioning (failed) in the complete system.

Definition 3.5: A **simple coherent decomposition of degree $(d_r)_{r=1,..,m}$** of (C,Φ) **with respect to the collection of its minimal path (cut) sets** is a coherent decomposition of degree $(d_r)_r$ of (C,Φ) all the elements of which are some simple coherent subsystems of (C,Φ) with respect to the collection of its minimal path (cut) sets.

Convention 3.6: Let $D = \{r \in \mathbf{N}^* \;/\; r \leq d\}$. If $D \cap \bar{M} = \emptyset$ in definitions 3.2 and 3.4, C' can be defined as follows: $C' = D \cup \bar{M}$. For convenience but without loss of generality, this is assumed in what follows.

Proposition 3.7: Given a binary broad-sense coherent system, (C,Φ),

(i) $\{(M,\mu_r) \;/\; r=1,\dots,d\}$ is a (simple) coherent subsystem of degree d of (C,Φ) (with respect to $\mathcal{H}(\Phi)$ or $\mathcal{K}(\Phi)$) and with an organizing system (C',κ) if and only if $\{(M,\mu_r^D) \,/\, r=1,\ldots,d\}$ is a (simple) coherent subsystem of the same degree of (C,Φ^D) (with respect to $\mathcal{K}(\Phi^D)$ or $\mathcal{H}(\Phi^D)$, respectively) and with an organizing system (C',κ^D).

(ii) $\{\{(M_r,\mu_{rv_r}) \,/\, v_r=1,\ldots,d_r\} \,/\, r=1,\ldots,m\}$ is a (simple) coherent decomposition of degree $(d_r)_r$ of (C,Φ) (with respect to $\mathcal{H}(\Phi)$ or $\mathcal{K}(\Phi)$) and with an organizing system (C',κ) if and only if $\{\{(M_r,\mu_{rv_r}^D) \,/\, v_r=1,\ldots,d_r\} \,/\, r=1,\ldots,m\}$ is a (simple) coherent decomposition of the same degree of (C,Φ^D) (with respect to $\mathcal{K}(\Phi^D)$ or $\mathcal{H}(\Phi^D)$, respectively) and with an organizing system (C',κ^D).

The **proof** is omitted since this result follows immediately from the concerned definitions (see (2.2)): as an indication, for every $\underline{x}\in S_C$, $\Phi^D(\underline{x}) = \kappa^D((\mu_r^D(\underline{x}_M))_{r=1,\ldots,d}, \underline{x}_{\overline{M}})$.

Furthermore, when restricted to the strict (simple) coherent subsystems and decompositions, the equivalence relations above remain valid: the strict-sense coherence is conserved by duality.

The next proposition allows to appraise the great generality of the simple coherent subsystems and decompositions. First, some conventions are needed.

Conventions 3.8: Some natural conventions allow to take into account any particular case included in the results and the proofs proposed in the following, without any tedious digression: given a finite non-empty set I, let D be a subset of I; if $D = \emptyset$, then, for every $\underline{x}\in S_I$, let $\coprod_{i\in D} x_i = 0$ and $\prod_{i\in D} x_i = 1$.

In addition, given some binary systems, (A,α_i), $i\in I$, for some finite non-empty set I, let $\mathcal{J}(((A,\alpha_i))_{i\in I}) = \{(A,\alpha_j) \,/\, j\in J\}$ for some $J \subset I$ which satisfies both of the following conditions:

(1) for every $(i,j)\in J^2$, $i \neq j \implies \alpha_i \not\equiv \alpha_j$;

(2) $\bigcup_{i\in I} \{(A,\alpha_i)\} = \bigcup_{j\in J} \{(A,\alpha_j)\}$.

Proposition 3.9: A simple coherent subsystem (decomposition) of (C,Φ) with respect to the collection of its minimal path sets can always be deduced from a coherent subsystem (decomposition) of (C,Φ). The same is true with respect to the collection of the minimal cut sets of (C,Φ).

Proof: This result follows immediately from the previous definitions. With the assumptions stated in definition 3.2, let $\mathcal{H}(\kappa) = \{B_u \ / \ u=1,\ldots,h'\}$, $G = \{u\in\{1,\ldots,h'\} \ / \ B_u\cap(C'\setminus\bar{M})\neq\emptyset\}$ and for every $u\in G$, $G_u = B_u\cap(C'\setminus\bar{M})$ and $\alpha_u \equiv \prod_{r\in G_u} \mu_r$. Then, for every $\underline{x}\in S_C$, $\kappa((\mu_r(\underline{x}_M))_{r=1,\ldots,d},\underline{x}_{\bar{M}}) = (\coprod_{u\in G} (\alpha_u(\underline{x}_M) . \prod_{i\in B_u\cap\bar{M}} x_i)) \vee (\coprod_{u\in\bar{G}} \prod_{i\in B_u} x_i)$.

Then, it appears immediately that $\mathcal{J}(((M,\alpha_u))_{u\in G})$ is a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$ (see Definition 3.4). Furthermore, the transformation defined above can be easily generalized for the coherent decompositions of (C,Φ).

In addition, the existence of a simple coherent subsystem (decomposition) with respect to $\mathcal{K}(\Phi)$ can be easily deduced, by duality (see Proposition 3.7) □

Henceforth, this study can be restricted to the simple coherent subsystems without yielding any restriction on its application domain: the results then obtained can be easily extended with the transformation defined in the proof proposed for proposition 3.9.

The next result plays a central part in what follows.

Theorem 3.10: Let $D = \{r\in\mathbf{N}^* \ / \ r\leq d\}$; let $\mathcal{M} = \{(M,\mu_r) \ / \ r\in D\}$ be a simple coherent subsystem of degree d of some binary coherent system, (C,Φ), with respect to the collection of its minimal path (cut) sets and with some organizing system $(C'=D\cup\bar{M},\kappa)$. For every non-empty subset A of C, if A is a minimal path (cut) set of (C,Φ) then, it satisfies one of the two following mutually exclusive conditions:

(3.2) either $A\cap M = \emptyset$ and A is a minimal path (cut) set of (C',κ);

(3.3) or $A\cap M$ is a minimal path (cut) set of (M,μ_r) for some $r\in D$ while $\{r\}\cup(A\cap\bar{M})$ is a minimal path (cut) set of (C',κ).

Conversely, if A satisfies (3.2), then A is a minimal path (cut) set of (C,Φ); if A satisfies (3.3), then A is a path (cut) set of (C,Φ) which is not necessarily minimal.

Proof: Let $\mathcal{M}$ be a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$. For every $H\in\mathcal{H}(\Phi)$, two cases can be distinguished.

Case 1: $H\cap M = \emptyset$; then, by (3.1), $\underline{z} = (\underline{1}_H,\underline{0}_D,\underline{0}_{\bar{H}\cap\bar{M}})\in S_{C'}$ verifies the following relation: $\kappa(\underline{z}) = \Phi(\underline{1}_H,\underline{0}_{\bar{H}}) = 1$; in addition, for every

$\underline{y} \in S_{C'}$, $\underline{y} < \underline{z}$ if and only if $(\underline{0}_M, \underline{y}_{\bar{M}}) < (\underline{1}_H, \underline{0}_{\bar{H}})$; consequently, by (3.1), $\kappa(\underline{y}) = \Phi(\underline{0}_M, \underline{y}_{\bar{M}}) = 0$ and $H \in \mathcal{H}(\kappa)$.

Conversely, by (3.1), if $A \subset C$ satisfies (3.2), then, $\Phi(\underline{y}) = \kappa(\underline{1}_A, \underline{0}_D, \underline{0}_{\bar{A}\cap\bar{M}}) = 1$, where $\underline{y} = (\underline{1}_A, \underline{0}_M, \underline{0}_{\bar{A}\cap\bar{M}})$; in addition, for every $\underline{x} \in S_C$, $\underline{x} < \underline{y}$ if and only if $(\underline{0}_D, \underline{x}_{\bar{M}}) < (\underline{1}_A, \underline{0}_D, \underline{0}_{\bar{A}\cap\bar{M}})$; consequently, by (3.1), $\Phi(\underline{x}) = \kappa(\underline{0}_D, \underline{x}_{\bar{M}}) = 0$ and $A \in \mathcal{H}(\Phi)$.

Case 2: $H \cap M \neq \emptyset$; then, by (3.1), $\underline{z} = ((\mu_r(\underline{1}_{H\cap M}, \underline{0}_{\bar{H}\cap M}))_{r\in D}, \underline{1}_{H\cap\bar{M}}, \underline{0}_{\bar{H}\cap\bar{M}})$ verifies the following relation: $\kappa(\underline{z}) = \Phi(\underline{1}_H, \underline{0}_{\bar{H}}) = 1$ and there must exist some $B \in \mathcal{H}(\kappa)$ such that $(\underline{1}_B, \underline{0}_{\bar{B}}) \leq \underline{z}$; consequently, $B \cap \bar{M} \subset H \cap \bar{M}$; indeed, the equality must hold: note $(\underline{1}_B, \underline{0}_{\bar{B}}) \leq (\underline{z}_D, \underline{1}_{B\cap\bar{M}}, \underline{0}_{\bar{B}\cap\bar{M}})$; so, $\Phi(\underline{1}_{H\cap M}, \underline{0}_{\bar{H}\cap M}, \underline{1}_{B\cap\bar{M}}, \underline{0}_{\bar{B}\cap\bar{M}}) = \kappa(\underline{z}_D, \underline{1}_{B\cap\bar{M}}, \underline{0}_{\bar{B}\cap\bar{M}}) = 1$ and $(H\cap M)\cup(B\cap\bar{M})$ is a path set of (C,Φ); therefore, $B\cap\bar{M} \subsetneq H\cap\bar{M}$ is inconsistent with $H \in \mathcal{H}(\Phi)$ and $B\cap\bar{M}$ must be equal to $H\cap\bar{M}$.

Let $D' = \{r \in D \ / \ \mu_r(\underline{1}_{H\cap M}, \underline{0}_{\bar{H}\cap\bar{M}}) = 1\}$; then, it can be easily checked that $B \cap D' \neq \emptyset$ by contraposition: if $B \cap D' = \emptyset$, then, $B \cap D = \emptyset$ and by (3.1), $\kappa(\underline{1}_B, \underline{0}_{\bar{B}}) = \Phi(\underline{0}_M, \underline{1}_{H\cap\bar{M}}, \underline{0}_{\bar{H}\cap\bar{M}}) = 1$; so, $H\cap\bar{M}$ would be a path set of (C,Φ), which is inconsistent with $H \in \mathcal{H}(\Phi)$; so, $B = \{r\}\cup(H\cap\bar{M})$ (see Definition 3.4) and $\mu_r(\underline{1}_{H\cap M}, \underline{0}_{\bar{H}\cap M}) = 1$, for some $r \in D'$.

In addition, it can be readily checked that $H\cap M \in \mathcal{H}(\mu_r)$ by contraposition: assume there exists some $H_0 \in \mathcal{H}(\mu_r)$ such that $H_0 \subsetneq H\cap M$; then, $H_0 \cup (H\cap\bar{M}) \subsetneq H$ and $H_0\cup(H\cap\bar{M})$ is a path set of (C,Φ), which is inconsistent with $H \in \mathcal{H}(\Phi)$.

Conversely, let A satisfy (3.3) for some $r \in D$; note

$(1_r, \underline{0}_{D\setminus\{r\}}, \underline{1}_{A\cap\bar{M}}, \underline{0}_{\bar{A}\cap\bar{M}}) \leq ((\mu_s(\underline{1}_{A\cap M}, \underline{0}_{\bar{A}\cap M}))_{s\in D}, \underline{1}_{A\cap\bar{M}}, \underline{0}_{\bar{A}\cap\bar{M}})$;

so, by (3.1), $\Phi(\underline{1}_A, \underline{0}_{\bar{A}}) = 1$ and A is a path set of (C,Φ) which is not necessarily minimal: this can be checked with the simple example proposed hereafter.

First, note the result with respect to $\mathcal{K}(\Phi)$ follows immediately by duality (see Proposition 3.7) □

Example 3.11: Let (C,Φ) be the binary strict-sense coherent system of order 5 defined as follows: let $C = \{i \in \mathbf{N}^* \ / \ i \leq 5\}$ and let $M = C\setminus\{3\}$; for every $\underline{x} \in S^5$, $\Phi(\underline{x}) = (\mu_1(\underline{x}_M).x_3) \vee \mu_2(\underline{x}_M)$ with $\mu_1(\underline{x}_M) = x_1x_5 \vee x_2x_4 \vee x_1x_4$ and $\mu_2(\underline{x}_M) = x_1x_4 \vee x_2x_5$.

It can be easily checked that $\{(M,\mu_r) \ / \ r=1,2\}$ is a simple coherent subsystem of degree 2 of (C,Φ) with respect to $\mathcal{K}(\Phi)$ and with an organizing system $(C'=\{1,2,3\},\kappa)$ defined as follows:

for every $\underline{y}\in S^3$, $\kappa(\underline{y}) = y_1y_3 \vee y_2$.

Indeed, (C,Φ) is the bridge system of order 5 [2; p.9]:

$\mathcal{H}(\Phi) = \{\{1,4\};\{2,5\};\{1,3,5\};\{2,3,4\}\}$.

Let $A = \{1,3,4\}$; $A\cap M = \{1,4\}\in\mathcal{H}(\mu_1)$ and $\{1\}\cup(A\cap\bar{M}) = \{1,3\}\in\mathcal{H}(\kappa)$; so, A satisfies (3.3) and by theorem 3.10, A is a path set of (C,Φ); but it is not minimal since $\{1,4\}\subset A$.

Remark 3.12: With the same assumptions as in theorem 3.10,

$$(3.4)\quad \mathcal{H}(\Phi)\subset\{H\in\mathcal{H}(\kappa)/H\subset\bar{M}\}\cup\Big(\bigcup_{r=1}^{d}\{H\cup H'/H\in\mathcal{H}(\mu_r);\{r\}\cup H'\in\mathcal{H}(\kappa)\}\Big)$$

$$\Big(\mathcal{K}(\Phi)\subset\{K\in\mathcal{K}(\kappa)/K\subset\bar{M}\}\cup\Big(\bigcup_{r=1}^{d}\{K\cup K'/K\in\mathcal{K}(\mu_r);\{r\}\cup K'\in\mathcal{K}(\kappa)\}\Big)\Big).$$

Conversely, $\{H\in\mathcal{H}(\kappa)\ /\ H\subset\bar{M}\}\subset\mathcal{H}(\Phi)$ $\quad(\{K\in\mathcal{K}(\kappa)\ /\ K\subset\bar{M}\}\subset\mathcal{K}(\Phi))$.

According to example 3.11, some further conditions are needed for ensuring the equality in (3.4). The conditions which are strictly necessary are put in evidence with the next corollary and lead to the pseudo-modules.

3.2. Minimal Path Sets and Cut Sets Through Pseudo-Modules:

Corollary 3.13: A simple coherent subsystem, $\mathcal{M}=\{(M,\mu_r)\ /\ r\in D\}$ of some binary coherent system, (C,Φ), with respect to the collection of its minimal path (cut) sets, can ensure the equality in (3.4) if and only if the minimal path (cut) sets of the corresponding organizing system, $(C'=D\cup\bar{M},\kappa)$, satisfy the following condition:

(3.5) for every $(r,s)\in D^2$ such that $r\neq s$, for every $(A,A')\in\mathcal{H}(\mu_r)\times\mathcal{H}(\mu_s)$ $\quad((A,A')\in\mathcal{K}(\mu_r)\times\mathcal{K}(\mu_s))$ and for every $(B,B')\in\mathcal{H}(\kappa)^2$ $((B,B')\in\mathcal{K}(\kappa)^2)$ such that $(r,s)\in B\times B'$, $A\cup(B\cap\bar{M})$ is not strictly included in $A'\cup(B'\cap\bar{M})$.

Proof: Let $\mathcal{M}$ be a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$. First, that (3.5) is necessary for ensuring the equality in (3.4) can be easily shown by contraposition. Assume there exist some $(r,s)\in D^2$, some $(A,A')\in\mathcal{H}(\mu_r)\times\mathcal{H}(\mu_s)$ and some $(B,B')\in\mathcal{H}(\kappa)^2$ such that: $r\neq s$, $(r,s)\in B\times B'$ and $H = A\cup(B\cap\bar{M})$ is strictly included in $H' = A'\cup(B'\cap\bar{M})$; now, H and H' satisfy (3.3); consequently, by theorem 3.10, both of them are some path sets of (C,Φ); so, the assumptions cannot ensure the equality in (3.4).

Conversely, assume $\mathcal{M}$ satisfies (3.5); let $A \subset C$; according to theorem 3.10, it suffices to show that if A satisfies (3.3), then, $A \in \mathcal{H}(\Phi)$. Indeed, A is a path set of (C,Φ), by theorem 3.10; so, it suffices to show that A is minimal; this can be done by contraposition. Assume there exists some $H \in \mathcal{H}(\Phi)$ such that $H \subsetneqq A$.

If $H \cap M = \emptyset$, then, by theorem 3.10, $H \cap \bar{M} \in \mathcal{H}(\varkappa)$ (see (3.2)), which is inconsistent with $\{r\} \cup (A \cap \bar{M}) \in \mathcal{H}(\varkappa)$ (see (3.3)) since $H \cap \bar{M} \subset A \cap \bar{M}$; consequently, H must intersect M and by theorem 3.10, $H \cap M \in \mathcal{H}(\mu_s)$, for some $s \in D$. If $r = s$, $H \cap M$ must be equal to $A \cap M$ since both of them are some minimal path sets of (M,μ_r) and $\{r\} \cup (H \cap \bar{M})$ must be equal to $\{r\} \cup (A \cap \bar{M})$ since both of them are some minimal path sets of $(C',\varkappa)$; but these equalities are inconsistent with the assumption, $H \subsetneqq A$. Therefore, r must be different from s; $(H \cap M, A \cap M) \in \mathcal{H}(\mu_s) \times \mathcal{H}(\mu_r)$, $(\{s\} \cup (H \cap \bar{M}), \{r\} \cup (A \cap \bar{M})) \in \mathcal{H}(\varkappa)^2$ and $H \subsetneqq A$, which is inconsistent with (3.5). So, A must be a minimal path set of (C,Φ) and the equality must hold in (3.4).

The corresponding result with respect to $\mathcal{K}(\Phi)$ follows immediately by duality (see Proposition 3.7) $\square$

Definition 3.14: A **pseudo-module of degree d** of (C,Φ) **with respect to the collection of its minimal path (cut) sets** is a simple coherent subsystem of degree d of (C,Φ) with respect to the same collection of sets, $\{(M,\mu_r) \ / \ r=1,\ldots,d\}$, such that the minimal path (cut) sets of the corresponding organizing system (defined by (3.1)), $(C'=\{i_r \in \mathbf{N}^* \ / \ i_r \notin \bar{M};\ r=1,\ldots,d\} \cup \bar{M}, \varkappa)$, satisfy the following condition:

(3.6) for every $(r,s) \in \{u \in \mathbf{N}^* \ / \ u \leq d\}^2$, $r \neq s$, if there exists some $(A,A') \in \mathcal{H}(\mu_r) \times \mathcal{H}(\mu_s)$ $((A,A') \in \mathcal{K}(\mu_r) \times \mathcal{K}(\mu_s))$ such that $A \subset A'$, then, for every $(B,B') \in \mathcal{H}(\varkappa)^2$ $((B,B') \in \mathcal{K}(\varkappa)^2)$ such that $(i_r,i_s) \in B \times B'$, $B \cap \bar{M}$ is not included in or equal to $B' \cap \bar{M}$.

Then, M is a **pseudo-modular subset of degree d** of (C,Φ) with respect to the collection of its minimal path (cut) sets.

A **strict pseudo-module** of (C,Φ) is defined with a strict simple coherent subsystem of (C,Φ).

Definition 3.15: A **pseudo-modular decomposition** of degree $(d_r)_{r=1,..,m}$ of (C,Φ) **with respect to the collection of its minimal path (cut) sets** is a coherent decomposition of degree $(d_r)_r$ of (C,Φ) all the elements of which are some pseudo-modules of (C,Φ) with respect to the collection of its minimal path (cut) sets.
A **strict pseudo-modular decomposition** of (C,Φ) is defined with a strict coherent decomposition of (C,Φ).

The next result follows immediately from (2.2) and proposition 3.7.

Proposition 3.16: The relations of duality stated in proposition 3.7 for the simple coherent subsystems and the simple coherent decompositions remain valid for the pseudo-modules and the pseudo-modular decompositions, respectively.

The next result follows from corollary 3.13 and extends some well-known relations for modules [3].Indeed, the condition (3.6) is somewhat stronger than (3.5) and the pseudo-modules allow to avoid any duplication of the minimal path (cut) sets in the right-hand term of the inclusion (3.4).

Corollary 3.17: Let $\{(M,\mu_r)\ /\ r=1,\ldots,d\}$ be a pseudo-module of degree d of some binary coherent system, (C,Φ), with respect to the collection of its minimal path (cut) sets and with an organizing system (C',κ); then,

(a) $$\mathcal{H}(\Phi) = \{H\in\mathcal{H}(\kappa)/H\subset\bar{M}\}\cup\left(\bigcup_{r=1}^{d}\{H\cup H'/H\in\mathcal{H}(\mu_r);\{r\}\cup H'\in\mathcal{H}(\kappa)\}\right)$$

$$(\mathcal{K}(\Phi) = \{K\in\mathcal{K}(\kappa)/K\subset\bar{M}\}\cup\left(\bigcup_{r=1}^{d}\{K\cup K'/K\in\mathcal{K}(\mu_r);\{r\}\cup K'\in\mathcal{K}(\kappa)\}\right));$$

(b) for every $r = 1, \ldots, d$,

$$\mathcal{H}(\mu_r) = \{H\cap M\neq\emptyset\ /\ H\in\mathcal{H}(\Phi):\{r\}\cup(H\cap\bar{M})\in\mathcal{H}(\kappa)\}$$

$$(\mathcal{K}(\mu_r) = \{K\cap M\neq\emptyset\ /\ K\in\mathcal{K}(\Phi):\{r\}\cup(K\cap\bar{M})\in\mathcal{K}(\kappa)\});$$

(c) $$\mathcal{H}(\kappa) = \{H\in\mathcal{H}(\Phi)/H\subset\bar{M}\}\cup\left(\bigcup_{r=1}^{d}\{\{r\}\cup(H\cap\bar{M})/H\in\mathcal{H}(\Phi):H\cap M\in\mathcal{H}(\mu_r)\}\right)$$

$$(\mathcal{K}(\kappa) = \{K\in\mathcal{K}(\Phi)/K\subset\bar{M}\}\cup\left(\bigcup_{r=1}^{d}\{\{r\}\cup(K\cap\bar{M})/K\in\mathcal{K}(\Phi):K\cap M\in\mathcal{K}(\mu_r)\}\right));$$

The next corollary follows immediately.

Corollary 3.18: Let $\{\{(M_r,\mu_{rv_r})\ /\ v_r=1,\ldots,d_r\}\ /\ r=1,\ldots m\}$ be a pseudo-modular decomposition of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ and with an organizing system (C',κ); then,

$$\mathcal{H}(\Phi) = \{\bigcup_{(r,v_r)\in H'} H_{rv_r} / H'\in\mathcal{H}(\kappa); H_{rv_r}\in\mathcal{H}(\mu_{rv_r}), v_r=1,..,d_r, r=1,..,m\}$$

$$(\mathcal{K}(\Phi) = \{\bigcup_{(r,v_r)\in K'} K_{rv_r} / K'\in\mathcal{K}(\kappa); K_{rv_r}\in\mathcal{K}(\mu_{rv_r}), v_r=1,..,d_r, r=1,..,m\}).$$

Once again, no restriction but only some refinement has been introduced with the pseudo-modules: this is shown with the next proposition.

Proposition 3.19: A pseudo-module (pseudo-modular decomposition) of (C,Φ) with respect to the collection of its minimal path sets can always be deduced from a coherent subsystem (coherent decomposition) of (C,Φ). The same is true with respect to the collection of the minimal cut sets of (C,Φ).

Proof: According to proposition 3.9, it suffices to show that a pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ can always be deduced from a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$. Indeed, this can be done by reiterating the transformation defined hereafter until pseudo-modularity is verified.

Let $\mathcal{M} = \{(M,\mu_r)\ /\ r\in D\}$ be a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ and with an organizing system $(C'=D\cup\bar{M},\kappa)$. Assume there exist some $(r,s)\in D^2$, some $(A,A')\in\mathcal{H}(\mu_r)\times\mathcal{H}(\mu_s)$ $((A,A')\in\mathcal{K}(\mu_r)\times\mathcal{K}(\mu_s))$ and some $(B,B')\in\mathcal{H}(\kappa)^2$ $((B,B')\in\mathcal{K}(\kappa)^2)$ such that: $r\neq s$, $(r,s)\in B\times B'$, $A\subset A'$ and $B\cap\bar{M}\subset B'\cap\bar{M}$. For convenience but without loss of generality, (r,s), (A,A') and (B,B') are assumed to be unique. Then, $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi)) = \mathcal{A}\setminus\{A'\cup(B'\cap\bar{M})\}$, where $\mathcal{A}$ coincides with the right-hand term of the inclusion (3.4) (see Corollary 3.13 and its proof). Two cases can be distinguished at first for determining a pseudo-module, $\mathcal{M}_1$ or $\mathcal{M}_2$, of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$.

Case 1: $|\{H\in\mathcal{H}(\kappa)\ /\ s\in H\}| = 1$ $(|\{K\in\mathcal{K}(\kappa)\ /\ s\in K\}| = 1)$; then, let

$\mathcal{M}_1 = (\mathcal{M}\setminus\{(M,\mu_s)\})\cup\{(M,\mu'_s)\ /\ \mu'_s\not\equiv 0\ (\mu'_s\not\equiv 1)\}$

with $\mathcal{H}(\mu'_s) = \mathcal{H}(\mu_s)\setminus\{A'\}$ if $|\mathcal{H}(\mu_s)| > 1$, $\mu'_s\equiv 0$, otherwise.

$(\mathcal{K}(\mu'_s) = \mathcal{K}(\mu_s)\setminus\{A'\}$ if $|\mathcal{K}(\mu_s)| > 1$, $\mu'_s\equiv 1$, otherwise).

$\mathcal{M}_1$ is a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$: as an indication, for every $\underline{x}\in S_C$,

$$\Phi(\underline{x}) = \begin{cases} \kappa(\mu'_s(\underline{x}_M),(\mu_u(\underline{x}_M))_{u\in D\setminus\{s\}},\underline{x}_{\bar{M}}) & \text{if } \mu'_s \not\equiv 0 \ (\mu'_s \not\equiv 1) \\ \alpha((\mu_u(\underline{x}_M))_{u\in D\setminus\{s\}},\underline{x}_{\bar{M}}) & \text{otherwise,} \end{cases}$$

where the binary coherent system $(C''=C'\setminus\{s\},\alpha)$ is defined as follows: $\mathcal{H}(\alpha) = \mathcal{H}(\kappa)\setminus\{B'\}$ $(\mathcal{K}(\alpha) = \mathcal{K}(\kappa)\setminus\{B'\})$.

Case 2: $|\{H\in\mathcal{H}(\kappa)\ /\ s\in H\}| > 1$ $(|\{K\in\mathcal{K}(\kappa)\ /\ s\in K\}| > 1)$; then, the system (C'',α) must be defined, in a more general way, as follows:

(3.7) $\mathcal{H}(\alpha) = \mathcal{H}(\kappa)\setminus\{H\in\mathcal{H}(\kappa)\ /\ s\in H\}$ $(\mathcal{K}(\alpha) = \mathcal{K}(\kappa)\setminus\{K\in\mathcal{K}(\kappa)\ /\ s\in K\})$;

let $\mathcal{M}_2 = \mathcal{M}_1\cup\{(M,\mu_{d+1})\ /\ \mu_{d+1}\not\equiv\mu_r\}$ with $\mathcal{H}(\mu_{d+1}) = \{A'\}$ $(\mathcal{K}(\mu_{d+1}) = \{A'\})$. So, $\mu_{d+1} \equiv \mu_r$ if and only if $|\mathcal{H}(\mu_r)| = 1$ $(|\mathcal{K}(\mu_r)| = 1)$ and $A = A'$.

If $\mu_{d+1} \not\equiv \mu_r$, then, for every $\underline{x}\in S_C$,

$$\Phi(\underline{x}) = \begin{cases} \beta(\mu_{d+1}(\underline{x}_M),\mu'_s(\underline{x}_M),(\mu_u(\underline{x}_M))_{u\in D\setminus\{s\}},\underline{x}_{\bar{M}}) & \text{if } \mu'_s \not\equiv 0 \ (\mu'_s \not\equiv 1) \\ \beta'(\mu_{d+1}(\underline{x}_M),(\mu_u(\underline{x}_M))_{u\in D\setminus\{s\}},\underline{x}_{\bar{M}}) & \text{otherwise,} \end{cases}$$

where the binary coherent systems $(C'\cup\{d+1\},\beta)$ and $(C''\cup\{d+1\},\beta')$ can be defined as follows (see (3.7)): $\mathcal{H}(\beta) = \mathcal{H}(\kappa)\cup\mathcal{E}$ and $\mathcal{H}(\beta') = \mathcal{H}(\alpha)\cup\mathcal{E}$ $(\mathcal{K}(\beta) = \mathcal{K}(\kappa)\cup\mathcal{E}$ and $\mathcal{K}(\beta') = \mathcal{K}(\alpha)\cup\mathcal{E})$ for $\mathcal{E} = \{\{d+1\}\cup H/\{s\}\cup H\in\mathcal{H}(\kappa);H\neq B'\cap\bar{M}\}$ $(\mathcal{E} = \{\{d+1\}\cup K/\{s\}\cup K\in\mathcal{K}(\kappa);K\neq B'\cap\bar{M}\})$.

If $\mu_{d+1} \equiv \mu_r$, then, for every $\underline{x}\in S_C$,

$$\Phi(\underline{x}) = \begin{cases} \gamma(\mu'_s(\underline{x}_M),(\mu_u(\underline{x}_M))_{u\in D\setminus\{s\}},\underline{x}_{\bar{M}}) & \text{if } \mu'_s \not\equiv 0 \ (\mu'_s \not\equiv 1) \\ \gamma'((\mu_u(\underline{x}_M))_{u\in D\setminus\{s\}},\underline{x}_{\bar{M}}) & \text{otherwise,} \end{cases}$$

where the binary coherent systems (C',γ) and (C'',γ') can be defined as follows (see (3.7)): $\mathcal{H}(\gamma) = \mathcal{H}(\kappa)\cup\mathcal{E}$ and $\mathcal{H}(\gamma') = \mathcal{H}(\alpha)\cup\mathcal{E}$ $(\mathcal{K}(\gamma) = \mathcal{K}(\kappa)\cup\mathcal{E}$ and $\mathcal{K}(\gamma') = \mathcal{K}(\alpha)\cup\mathcal{E})$ for

$\mathcal{E} = \{\{r\}\cup H\ /\ \{s\}\cup H\in\mathcal{H}(\kappa);H\neq B'\cap\bar{M}\}$ $(\mathcal{E} = \{\{r\}\cup K\ /\ \{s\}\cup K\in\mathcal{K}(\kappa);K\neq B'\cap\bar{M}\})$.

In addition, the pseudo-modularity of $\mathcal{M}_1$ or of $\mathcal{M}_2$ now appears immediate (see (3.6)) $\square$

Example 3.20: A pseudo-module $\mathcal{M}$ of (C,Φ) with respect to $\mathcal{H}(\Phi)$ can be easily determined from the simple coherent subsystem defined in example 3.11: let $\mathcal{H}(\alpha_1) = \mathcal{H}(\mu_1)\setminus\{\{1,4\}\}$ and $\alpha_2 \equiv \mu_2$; then, $\mathcal{M} = \{(M,\alpha_r)\ /\ r=1,2\}$ and its organizing system is equal to (C',κ).

Remark 3.21: Indeed, a pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ can be considered as a coherent subsystem with respect to $\mathcal{K}(\Phi)$ $(\mathcal{H}(\Phi))$, only (see Example 3.20). But, by proposition 3.19, a pseudo-module of (C,Φ) with respect to $\mathfrak{K}(\Phi)$ can always be deduced from a pseudo-module of (C,Φ) with respect to $\mathfrak{H}(\Phi)$ and conversely.

3.3. Pseudo-Modules Versus Modules; Some Simple Examples:

Remark 3.22: That pseudo-modules are much more general than modules appears immediately: a module (modular decomposition) of (C,Φ) is a pseudo-module (pseudo-modular decomposition) of degree 1 ($\underline{1}\in S^m$) of (C,Φ), both with respect to $\mathfrak{H}(\Phi)$ and $\mathfrak{K}(\Phi)$. Consequently, any "trivial module" [3] (i.e. defined from a singleton of C relevant to (C,Φ) or from a subset of C including the set of the components relevant to (C,Φ)) is a trivial pseudo-module of (C,Φ) both with respect to $\mathfrak{H}(\Phi)$ and $\mathfrak{K}(\Phi)$.

In an alternative way, the great generality of pseudo-modules with respect to modules can be easily appraised by considering some simple systems which are well known for not admitting any module except the trivial ones.

Examples 3.23: This is the case of the "bridge system" of order 5 (see example 3.11). However, it includes a (strict) pseudo-modular decomposition of degree (2,1)

(i) with respect to the collection of its minimal path sets: $\{\mathcal{M}\}\cup\{((\{3\},\beta))\}$ where the pseudo-module $\mathcal{M}$ has been defined in example 3.20 while for every $x\in S$, $\beta(x) = x$;

(ii) with respect to the collection of its minimal cut sets: $\{\{(M,\mu_r)\ /\ r=1,2\};\{(\bar{M},\mu_3)\}\}$ where $\mu_1 \equiv \alpha_1^D$, $\mu_3 \equiv \beta$ and for every $\underline{x}_M\in S_M$, $\mu_2(\underline{x}_M) = (x_1 \vee x_2)(x_4 \vee x_5)$; the corresponding organizing system, $(\{1,2,3\},\kappa')$ relies upon a parallel-series structure: $\kappa' = \kappa^D$, where the structure function κ has been defined in example 3.11.

The bridge system has been equally considered in [7] in order to show that even if $\mathfrak{J}_M(\mathfrak{H}(\Phi))$ and $\mathfrak{J}_M(\mathfrak{K}(\Phi))$ satisfy the incomparableness conditions (2.3), they may not be the collections of the minimal path sets and of the minimal cut sets of the same binary coherent system and consequently, M may not be a modular subset for the concerned system (see Remark 2.14).

Examples 3.24: As it is well known, any k-out-of-n system [2] does not admit any module other than the trivial ones. However, such systems admit some pseudo-modules and even some pseudo-modular decompositions which are not trivial.

For instance, the following strict pseudo-modular decomposition of degree 3, $\mathcal{M}_1 = \{(C,\mu_r) \ / \ r=1,2,3\}$, can be easily detected with respect to the collection of the minimal path sets of the 2-out-of-4 system: for every $\underline{x} \in S^4$, $\mu_1(\underline{x}) = x_1x_2 \vee x_3x_4$, $\mu_2(\underline{x}) = x_1x_3 \vee x_2x_4$ and $\mu_3(\underline{x}) = x_1x_4 \vee x_2x_3$; of course, the corresponding organizing system, $(\{1,2,3\},\varkappa)$, relies upon a parallel-structure: for every $\underline{x} \in S^4$, $\varkappa((\mu_r(\underline{x}))_{r=1,2,3}) = \coprod_{r=1}^{3} \mu_r(\underline{x})$.

Another finer pseudo-modular decomposition of degree (2,2) with respect to the same collection can be defined as follows: let $M = \{1,2\}$ and $\mathcal{M}_2 = \{\{(M,\mu_{1s}) \ / \ s=1,2\};\{(\bar{M},\mu_{2s}) \ / \ s=1,2\}\}$, where for every $\underline{x} \in S_C$, $\mu_{11}(\underline{x}_M) = x_1x_2$, $\mu_{12}(\underline{x}_M) = x_1 \vee x_2$, $\mu_{21}(\underline{x}_{\bar{M}}) = x_3x_4$, and $\mu_{22}(\underline{x}_{\bar{M}}) = x_3 \vee x_4$. Its organizing system, $(C'=\{1,2\}^2,\varkappa)$, is immediate: for every $\underline{y} \in S^4$, $\varkappa(\underline{y}) = y_{11} \vee y_{12}y_{22} \vee y_{21}$.

In addition, it can be easily checked that $\mathcal{M}_2$ is also a strict pseudo-modular decomposition with respect to the collection of the minimal cut sets with an organizing system, $(C',\varkappa')$, defined as follows: for every $\underline{y} \in S^4$, $\varkappa'(\underline{y}) = (y_{11} \vee y_{22})(y_{12} \vee y_{21})$.

The next result extends the transitivity property verified by modules.

Proposition 3.25: The relation "to be a coherent subsystem of" as also the relations "to be a simple coherent subsystem of" and "to be a pseudo-module of", defined with respect to the collection of the minimal path (cut) sets, are transitive: as an indication, in terms of pseudo-modules, for any binary coherent system, (C,Φ), and for every pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$, $\mathcal{M} = \{(M,\mu_r) \ / \ r \in D\}$, if for some $L \subset M$, some $E \subset \mathbf{N}^*$, some covering $\{E_r \ / \ r \in D\}$ of E and some collection of (distinct) systems, $\{(L,\alpha_u) \ / \ u \in E\}$, the collection $\mathcal{A}_r = \{(L,\alpha_u) \ / \ u \in E_r\}$ is a pseudo-module of (M,μ_r) with respect to $\mathcal{H}(\mu_r)$ $(\mathcal{K}(\mu_r))$, $r \in D$ respectively, then, $\{(L,\alpha_u) \ / \ u \in E\}$ is a pseudo-module of degree $|E| \le \sum_{r \in D} |E_r|$ of

(C,Φ) with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$).

Proof: The result in terms of the coherent subsystems appears immediate: for every $\underline{x}\in S_C$,

$$\Phi(\underline{x}) = \kappa((\mu_r(\underline{x}_M))_{r\in D},\underline{x}_{\bar{M}}) = \kappa((\beta_r((\alpha_u(\underline{x}_L))_{u\in E_r},\underline{x}_{M\cap\bar{L}}))_{r\in D},\underline{x}_{\bar{M}})$$
$$= \alpha((\alpha_u(\underline{x}_L))_{u\in E},\underline{x}_{\bar{L}}),$$

for some binary coherent systems $(C'=D\cup\bar{M},\kappa)$ and $(E_r\cup(\bar{L}\cap M),\beta_r)$, $r\in D$, the organizing systems for the concerned coherent subsystems.

Therefore, if $\mathcal{M}$ and $\mathcal{A}_r$ are some simple coherent subsystems of (C,Φ) and of (M,μ_r), $r\in D$ respectively, with respect to the collection of their minimal path sets, then, it appears immediately that $\{(E\cup(\bar{L}\cap M),\beta_r)\ /\ r\in D\}$ is itself a simple coherent subsystem of $(C"=E\cup\bar{L},\alpha)$ with respect to $\mathcal{H}(\alpha)$; (C',κ) is its organizing system and by theorem 3.10 (see (3.4)),

$$(3.8)\quad \mathcal{H}(\alpha)\subset\{H\in\mathcal{H}(\kappa)\ /\ H\subset\bar{M}\}\ \cup\ (\bigcup_{r\in D}\{H\cup H'\ /\ H\in\mathcal{H}(\beta_r);\{r\}\cup H'\in\mathcal{H}(\kappa)\}).$$

So, for every $A\in\mathcal{H}(\alpha)$, $A\cap E$ is either empty or equal to $H\cap E$, for some $H\in\mathcal{H}(\beta_r)$ with some $r\in D$; now, $(E_r\cup\bar{L},\beta_r)$ is the organizing system of the simple coherent subsystem $\mathcal{A}_r$ of (M,μ_r) with respect to $\mathcal{H}(\mu_r)$; therefore, for every $A\in\mathcal{H}(\alpha)$, $|A\cap E|\leq 1$ and $\{(L,\alpha_r)\ /\ r\in E\}$ is a simple coherent subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$.

In addition, the pseudo-modularity of $\mathcal{M}$ and of $\mathcal{A}_r$, $r\in D$, ensures the same property for $\{(L,\alpha_u)\ /\ u\in E\}$. This can be easily checked by contraposition: assume there exists $(u,v)\in E^2$, $(A,A')\in\mathcal{H}(\alpha_u)\times\mathcal{H}(\alpha_v)$ and $(B,B')\in\mathcal{H}(\alpha)^2$ such that $u\neq v$, $A\subset A'$, $(u,v)\in B\times B'$ and $B\cap\bar{L}\subset B'\cap\bar{L}$; if $(u,v)\in E_r^2$ for some $r\in D$, then, by (3.8), $(\{u\}\cup(B\cap\bar{L}\cap M),\{v\}\cup(B\cap\bar{L}\cap M))\in\mathcal{H}(\beta_r)^2$ and $B\cap\bar{L}\cap M\subset B'\cap\bar{L}\cap M$; so, the assumptions are inconsistent with the pseudo-modularity of $\mathcal{A}_r$; if $(u,v)\in E_r\times E_s$ for some $(r,s)\in D^2$ such that $r\neq s$, then, by (3.8) and corollary 3.17, $(A\cup(B\cap\bar{L}\cap M),A'\cup(B'\cap\bar{L}\cap M))\in\mathcal{H}(\mu_r)\times\mathcal{H}(\mu_s)$; in addition, by (3.8), $(\{r\}\cup(B\cap\bar{M}),\{s\}\cup(B'\cap\bar{M}))\in\mathcal{H}(\kappa)^2$; but, $A\cup(B\cap\bar{L}\cap M)\subset A'\cup(B'\cap\bar{L}\cap M)$ and $B\cap\bar{M}\subset B'\cap\bar{M}$; so, the assumptions are inconsistent with the pseudo-modularity of $\mathcal{M}$. Consequently, $\{(L,\alpha_r)\ /\ r\in E\}$ must be a pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ and the corresponding result with respect to $\mathcal{K}(\Phi)$ follows immediately by duality (see Propositions 3.7 and 3.16) $\square$

Remark 3.26: As coherence (see (2.4)) and modules, the (simple) coherent subsystems and the pseudo-modules can be characterized in terms of life functions by extending to some larger domain and range their definitions in terms of structure functions. As an indication, with the assumptions stated in definition 3.3, let τ, τ_{rv_r} and τ' be the life functions of (C,Φ), (M,μ_{rv_r}), $(r,v_r)\in C'$, and (C',κ) respectively; for every $u\in\bar{R}^+$ (see Definition 2.16),

$$J_u\circ\tau = \Phi\circ\underline{J}_u = \kappa\circ((\mu_{rv_r}\circ\underline{J}_{uM_r})_{(r,v_r)\in C'})$$
$$= \kappa\circ((J_u\circ\tau_{rv_r})_{(r,v_r)\in C'}) = J_u\circ\tau'\circ(\tau_{rv_r})_{(r,v_r)\in C'},$$ where

for every $r = 1,\dots,m$ and every $\underline{t}\in\bar{R}^{+|C|}$, $J_{uM_r}(\underline{t}) = (J_u(t_i))_{i\in M_r}$.

4. GENERALIZED "NETWORK REPRESENTATIONS":

Both of the theorems proposed in this section yield some generalizations for the "network representations" proposed in [3] for the binary (strict-sense) coherent systems which contain a (strict) module. They are stated with conventions 3.8 and illustrated with figures 1 and 2.

Theorem 4.1: $\mathcal{M} = \{(M,\mu_r) \;/\; r\in D\}$, where $D = \{r\in\mathbf{N}^* \;/\; r\le d\}$, is a simple coherent subsystem (pseudo-module) of degree d of some binary broad-sense coherent system, (C,Φ), with respect to the collection of its minimal path sets, $\mathcal{H}(\Phi)$, if and only if there exist some binary semi-coherent systems, $(\bar{M},\nu)$ and $(\bar{M},\nu_r)$, $r\in D$, such that:
$\{\mathcal{M}\}\cup\{\mathcal{M}'=\{(\bar{M},\alpha)\in\mathcal{J}(((\bar{M},\nu_r))_{r\in D},(\bar{M},\nu))/\alpha\not\equiv 0;\ \alpha\not\equiv 1\} \;/\; \mathcal{M}'\neq\emptyset\}$
is a simple coherent decomposition (pseudo-modular decomposition) of degree (d,d') of (C,Φ) with respect to $\mathcal{H}(\Phi)$ and verifies the following relations: $d' \le d+1$ and

(4.1) $\forall \underline{x}\in S_C$, $\Phi(\underline{x}) = \nu(\underline{x}_{\bar{M}}) \vee \coprod_{r=1}^{d} \mu_r(\underline{x}_M)\,\nu_r(\underline{x}_{\bar{M}})$.

Furthermore, let (C',κ) be the organizing system for $\mathcal{M}$.
$(\bar{M},\nu)$ is only semi-coherent if and only if $\nu\equiv 0$, which in its turn is equivalent to the following condition: for every $B\in\mathcal{H}(\kappa)$, $B\cap D\neq\emptyset$ (for every $H\in\mathcal{H}(\Phi)$, $H\cap M\neq\emptyset$). For each $r\in D$, $(\bar{M},\nu_r)$ is only semi-coherent if and only if $\nu_r\equiv 1$, which in its turn is equivalent to the following condition: $\{r\}\in\mathcal{H}(\kappa)$ ($\mathcal{H}(\mu_r)\subset\mathcal{H}(\Phi)$).

Proof: To show that the existence of the simple coherent subsystem $\mathcal{M}$ implies the existence of some simple coherent decomposition of (C,Φ) with respect to $\mathcal{H}(\Phi)$, it suffices to consider the $(d+1)$ binary systems $(\bar{M},\nu_r)$, $r\in D$, and $(\bar{M},\nu)$ defined as follows:

(4.2) for every $r\in D$ and for every $\underline{x}_{\bar{M}}\in S_{\bar{M}}$,

$$\nu_r(\underline{x}_{\bar{M}}) = 1 \iff \exists B\in \mathcal{H}(\kappa) \ : \ r\in B \text{ and } (\underline{1}_{B\cap\bar{M}},\underline{0}_{\bar{B}\cap\bar{M}}) \leq \underline{x}_{\bar{M}}$$

$$\nu(\underline{x}_{\bar{M}}) = 1 \iff \exists B\in \mathcal{H}(\kappa) \ : \ B\cap D = \emptyset \text{ and } (\underline{1}_{B},\underline{0}_{\bar{B}\cap\bar{M}}) \leq \underline{x}_{\bar{M}} .$$

For every $\underline{x}\in S_C$, $\Phi(\underline{x}) = 1$ if and only if $(\underline{1}_H,\underline{0}_{\bar{H}}) \leq \underline{x}$, for some $H\in \mathcal{H}(\Phi)$; so, by theorem 3.10, there is some subset H of C such that:

(4.3) either $H\cap M = \emptyset$, $H\cap\bar{M}\in \mathcal{H}(\kappa)$ and $(\underline{1}_H,\underline{0}_{\bar{H}\cap\bar{M}}) \leq \underline{x}_{\bar{M}}$; or $H\cap M\in \mathcal{H}(\mu_r)$ and $\{r\}\cup(H\cap\bar{M})\in \mathcal{H}(\kappa)$, for some $r\in D$, while $(\underline{1}_H,\underline{0}_{\bar{H}}) \leq \underline{x}$.

By (4.2), this yields the following equality:

(4.4) $\nu(\underline{x}_{\bar{M}}) \vee \coprod_{r\in D} \mu_r(\underline{x}_M)\ \nu_r(\underline{x}_{\bar{M}}) = 1.$

Conversely, by (4.2), for every $\underline{x}\in S_C$, if (4.4) holds, (4.3) must equally hold for some $H\subset C$; and by theorem 3.10, if H satisfies (4.3), it is a path set of (C,Φ) such that $(\underline{1}_H,\underline{0}_{\bar{H}}) \leq \underline{x}$; consequently, $\Phi(\underline{x}) = 1$ and (4.1) holds.

The semi-coherence of $(\bar{M},\nu)$ is obvious: by (4.1),

(4.5) for every $\underline{x}_{\bar{M}}\in S_{\bar{M}}$, $\nu(\underline{x}_{\bar{M}}) = \Phi(\underline{0}_M,\underline{x}_{\bar{M}})$;

in addition, by (4.1), $\nu \equiv 1$ if and only if $\Phi \equiv 1$, which is inconsistent with the broad-sense coherence of (C,Φ). So, $(\bar{M},\nu)$ is only semi-coherent if and only if $\nu \equiv 0$ (see Remark 2.6), which by (4.2), is equivalent to the following condition: for every $B\in \mathcal{H}(\kappa)$,

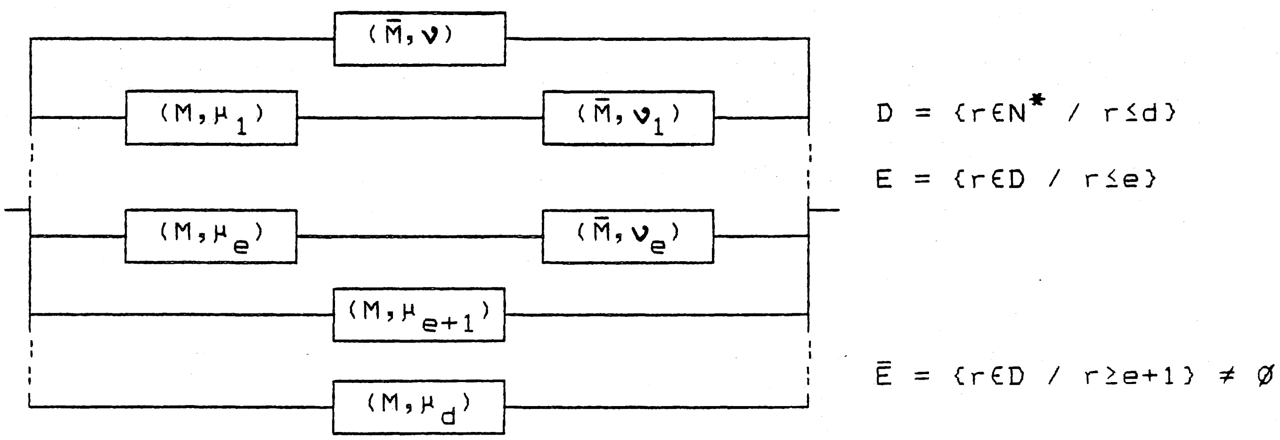


Figure 1: Representation of the Organizing system for a Simple Coherent Subsystem of (C,Φ) with respect to $\mathcal{H}(\Phi)$, $\{(M,\mu_r) \ / \ r\in D\}$ (see Theorem 4.1 and (4.8)).

$B \cap D \neq \emptyset$. But, by (4.5), if $\nu \not\equiv 0$, $(\bar{M},\nu)$ must be broad-sense coherent. For each $r \in D$ respectively, by (4.2), $(\bar{M},\nu_r)$ is semi-coherent; in addition, $\nu_r \equiv 0$ if and only if r is irrelevant to (C',κ), which is inconsistent with the degree exactly equal to d which has been assumed for the simple coherent subsystem $\mathcal{M}$; therefore, $(\bar{M},\nu_r)$ is only semi-coherent if and only if $\nu_r \equiv 1$ (see Remark 2.6), which by (4.2), is equivalent to the following condition: $\{r\} \in \mathcal{H}(\kappa)$. But, by (4.2), if $\nu_r \not\equiv 1$, then, $(\bar{M},\nu_r)$ must be broad-sense coherent.
Consequently, $\{\mathcal{A} \,/\, \mathcal{A} = \mathcal{M}$ or $\mathcal{M}'$; $\mathcal{M}' \neq \emptyset\}$ is a simple coherent decomposition of (C,Φ) of degree (d,d') with respect to $\mathcal{H}(\Phi)$ (see definition 3.4); obviously, $d' = |E|+1 \leq d+1$, where $E = \{r \in D \,/\, \nu_r \not\equiv 1\}$.
Concerning the particular case when $\mathcal{M}$ is a pseudo-module of (C,Φ), first, note (4.1) can be obtained directly by equivalence since corollary 3.17 then holds; indeed, $(\bar{M},\nu_r)$, $r \in D$, and $(\bar{M},\nu)$ then can be defined, in an alternative way, as follows (see (4.2)):

(4.6) for every $r \in D$ and for every $\underline{x}_{\bar{M}} \in S_{\bar{M}}$,

$$\nu_r(\underline{x}_{\bar{M}}) = 1 \iff \exists H \in \mathcal{H}(\Phi) : H \cap M \in \mathcal{H}(\mu_r) \text{ and } (\underline{1}_{H \cap \bar{M}}, \underline{0}_{\bar{H} \cap \bar{M}}) \leq \underline{x}_{\bar{M}};$$

$$\nu(\underline{x}_{\bar{M}}) = 1 \iff \exists H \in \mathcal{H}(\Phi) : H \cap M = \emptyset \text{ and } (\underline{1}_{H}, \underline{0}_{\bar{H} \cap \bar{M}}) \leq \underline{x}_{\bar{M}}.$$

In addition, that the pseudo-modularity of $\mathcal{M}$ ensures the same property for $\mathcal{M}'$ appears immediate: first, note it suffices to check that $\mathcal{M}'$ satisfies (3.6) for every couple $(r,s) \in E^2$ such that $\nu_r \not\equiv \nu_s$, only; let $(C''=D' \cup M, \alpha)$, for some $D' \subset \mathbf{N}^*$ such that $|D'| = d'$ and $D' \cap M = \emptyset$, be the organizing system for the simple coherent subsystem $\mathcal{M}'$; now, it suffices to note that for every $r \in E$, $A \in \mathcal{H}(\mu_r)$ $(B \in \mathcal{H}(\nu_{i_r}))$ if and only if $\{i_r\} \cup A \in \mathcal{H}(\alpha)$ $(\{r\} \cup B \in \mathcal{H}(\kappa))$, for some $i_r \in D'$ such that $\nu_{i_r} \equiv \nu_r$ (see (4.1)) $\square$

The next result follows immediately from theorem 4.1, by duality (see Propositions 3.7 and 3.16); as an indication, (4.1) is equivalent to the following relation: for every $\underline{x} \in S_C$,

$$\Phi^D(\underline{x}) = \nu^D(\underline{x}_{\bar{M}}) \;.\; \prod_{r=1}^{d} (\mu_r^D(\underline{x}_M) \vee \nu_r^D(\underline{x}_{\bar{M}})).$$

Theorem 4.2: $\mathcal{M} = \{(M,\mu_r) \,/\, r \in D\}$, where $D = \{r \in \mathbf{N}^* \,/\, r \leq d\}$, is a simple coherent subsystem (pseudo-module) of degree d of some binary broad-sense coherent system, (C,Φ), with respect to the collection of its minimal cut sets, $\mathcal{H}(\Phi)$, if and only if there exist some binary

semi-coherent systems, $(\bar{M},\nu)$ and $(\bar{M},\nu_r)$, $r\in D$, such that:
$\{\mathcal{M}\}\cup\{\mathcal{M}'=\{(\bar{M},\alpha)\in\mathcal{J}(((\bar{M},\nu_r))_{r\in D},(\bar{M},\nu))/\alpha\not\equiv 0;\ \alpha\not\equiv 1\}\ /\ \mathcal{M}'\neq\emptyset\}$
is a simple coherent decomposition (pseudo-modular decomposition) of degree (d,d') of (C,Φ) with respect to $\mathcal{K}(\Phi)$ and verifies the following relations: $d'\le d+1$ and

(4.7) $\forall\underline{x}\in S_C,\ \Phi(\underline{x}) = \nu(\underline{x}_{\bar{M}}).\ \prod_{r=1}^{d}(\mu_r(\underline{x}_M)\vee\nu_r(\underline{x}_{\bar{M}}))$.

Furthermore, let (C',κ) be the organizing system for $\mathcal{M}$.
$(\bar{M},\nu)$ is only semi-coherent if and only if $\nu\equiv 1$, which in its turn is equivalent to the following condition: for every $B\in\mathcal{K}(\kappa)$, $B\cap D\neq\emptyset$ (for every $K\in\mathcal{K}(\Phi)$, $K\cap M\neq\emptyset$). For each $r\in D$, $(\bar{M},\nu_r)$ is only semi-coherent if and only if $\nu_r\equiv 0$, which in its turn is equivalent to the following condition: $\{r\}\in\mathcal{K}(\kappa)$ $(\mathcal{K}(\mu_r)\subset\mathcal{K}(\Phi))$.

Remark 4.3: (4.1) ((4.7)) can be respectively re-stated as follows (see Figures 1 and 2): let $E=\{r\in D\ /\ \nu_r\not\equiv 1\}$ $(E=\{r\in D\ /\ \nu_r\not\equiv 0\})$;

(4.8) $\forall\underline{x}\in S_C,\ \Phi(\underline{x}) = \nu(\underline{x}_{\bar{M}})\vee(\coprod_{r\in E}\mu_r(\underline{x}_M)\ \nu_r(\underline{x}_{\bar{M}}))\vee(\coprod_{r\in\bar{E}}\mu_r(\underline{x}_M))$

$(\Phi(\underline{x}) = \nu(\underline{x}_{\bar{M}}).(\prod_{r\in E}(\mu_r(\underline{x}_M)\vee\nu_r(\underline{x}_{\bar{M}}))).(\prod_{r\in\bar{E}}\mu_r(\underline{x}_M)))$.

In addition, $(\bar{M},\nu)$ and $(\bar{M},\nu_r)$, $r\in E$, are uniquely defined from the simple coherent subsystem $\mathcal{M}$ of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ by the following relations (see (4.2)): if $\nu\not\equiv 0$ $(\nu\not\equiv 1)$,

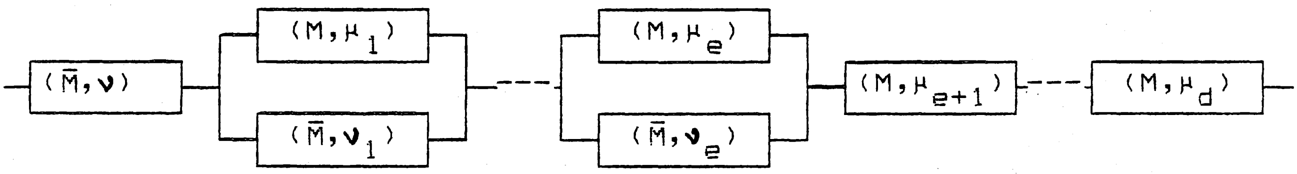
(4.9) $\mathcal{H}(\nu) = \{B\in\mathcal{H}(\kappa)\ /\ B\subset\bar{M}\}$; $\mathcal{H}(\nu_r) = \{B\cap\bar{M}\neq\emptyset\ /\ B\in\mathcal{H}(\kappa):r\in B\}$

$(\mathcal{K}(\nu) = \{B\in\mathcal{K}(\kappa)\ /\ B\subset\bar{M}\}$; $\mathcal{K}(\nu_r) = \{B\cap\bar{M}\neq\emptyset\ /\ B\in\mathcal{K}(\kappa):r\in B\})$.

And for the particular case when $\mathcal{M}$ is a pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$, by (4.6):

(4.10) $\mathcal{H}(\nu) = \{H\in\mathcal{H}(\Phi)/H\subset\bar{M}\}$; $\mathcal{H}(\nu_r) = \{H\cap\bar{M}\neq\emptyset/H\in\mathcal{H}(\Phi):H\cap M\in\mathcal{H}(\mu_r)\}$

$(\mathcal{K}(\nu) = \{K\in\mathcal{K}(\Phi)/K\subset\bar{M}\}$; $\mathcal{K}(\nu_r) = \{K\cap\bar{M}\neq\emptyset/K\in\mathcal{K}(\Phi):K\cap M\in\mathcal{K}(\mu_r)\})$.



$D=\{r\in\mathbf{N}^*\ /\ r\le d\}$ and $E=\{r\in D\ /\ r\le e\}$; $\bar{E}\neq\emptyset$.

Figure 2: Representation of the Organizing system for a Simple Coherent Subsystem of (C,Φ) with respect to $\mathcal{K}(\Phi)$, $\{(M,\mu_r)\ /\ r\in D\}$ (see Theorem 4.2 and (4.8)).

Notations 4.4: For any collection of subsets of C, $\mathcal{A}$, let $\mathcal{A}_M = \{A\in \mathcal{A} \ / \ A\cap M\neq\emptyset\}$ and $\mathcal{A}_{M\bar{M}} = \mathcal{A}_{M\cap\bar{M}} = \mathcal{A}_M\cap \mathcal{A}_{\bar{M}}$.
In particular, let $\mathcal{H}_M = \mathcal{H}(\Phi)_M$, $\mathcal{K}_M = \mathcal{K}(\Phi)_M$, $\mathcal{H}_{M\bar{M}} = \mathcal{H}_M\cap\mathcal{H}_{\bar{M}}$ and $\mathcal{K}_{M\bar{M}} = \mathcal{K}_M\cap\mathcal{K}_{\bar{M}}$.

Remark 4.5: An important consequence for theorems 4.1 and 4.2 is the following: any non-empty subset of C, relevant to (C,Φ), can be a pseudo-modular subset of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$. Indeed, by proposition 3.19, it suffices to check this result holds for the (simple) coherent subsets (see (4.8)): by (2.1) (see Remark 2.11 and Conventions 3.8), for every $\underline{x}\in S_C$,

$$\Phi(\underline{x}) = (\coprod_{r\in E_M}\eta_{r/M}(\underline{x}_M))\vee(\coprod_{r\in D_{M\cap\bar{M}}}\eta_{r/M}(\underline{x}_M)\eta_{r/\bar{M}}(\underline{x}_{\bar{M}}))\vee(\coprod_{r\in E_{\bar{M}}}\eta_{r/\bar{M}}(\underline{x}_{\bar{M}}))$$

$$(\Phi(\underline{x}) = (\prod_{r\in E_M}\theta_{r/M}(\underline{x}_M))(\prod_{r\in D_{M\cap\bar{M}}}(\theta_{r/M}(\underline{x}_M)\vee\theta_{r/\bar{M}}(\underline{x}_{\bar{M}})))(\prod_{r\in E_{\bar{M}}}\theta_{r/\bar{M}}(\underline{x}_{\bar{M}})))$$

where for every $A = M$, $\bar{M}$ or $M\cap\bar{M}$, $D_A = \{r=1,..,h \ / \ H_r\in\mathcal{H}_A\}$ $(D_A = \{r=1,..,k \ / \ K_r\in\mathcal{K}_A\})$; for every $A = M$ or $\bar{M}$, $E_A = D_A\setminus D_{M\cap\bar{M}}$ and $\eta_{r/A}$ $(\theta_{r/A})$ denotes the restriction of η_r (θ_r) to S_A, $r = 1, ...,h$ $(r = 1, ...,k)$ respectively.
In addition, if for every $(A,A')\in\mathcal{H}^2_{M\bar{M}}$ $((A,A')\in\mathcal{K}^2_{M\bar{M}})$ such that $A \neq A'$, $A\cap M \neq A'\cap M$, then, $\{(M,\eta_{r/M}) \ / \ r\in D_M\}$ $(\{(M,\theta_{r/M}) \ / \ r\in D_M\})$ is a pseudo-module of degree $|\mathcal{H}_M|$ $(|\mathcal{K}_M|)$ of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$. Such pseudo-modules are retained henceforth as the "trivial pseudo-modules" (see Remark 3.22).

Remark 4.6: Another consequence for theorems 4.1 and 4.2 is that the simple coherent subsystems as also the pseudo-modules generally cannot enjoy loneliness: as soon as a (simple coherent subsystem) pseudo-module is identified, a (simple coherent decomposition) pseudo-modular decomposition is determined.
When restricted to the modular subsets, this implication is false: some conditions are needed on $\mathcal{H}(\Phi)$ and $\mathcal{K}(\Phi)$ for ensuring the modularity of the complement of a modular subset. These conditions have been proved in [7] with some set arguments but they can be deduced, in a shorter way, from theorems 4.1 and 4.2: let M be a pseudo-modular subset of (C,Φ); M and $\bar{M}$ are two (pseudo-) modular subsets (of degree 1) of (C,Φ) if and only if their corresponding

modules verify one of the following mutually exclusive relations (see (4.8)): either $\forall \underline{x} \in S_C$, $\Phi(\underline{x}) = \mu(\underline{x}_M) \vee \nu(\underline{x}_{\bar{M}})$ or $\forall \underline{x} \in S_C$, $\Phi(\underline{x}) = \mu(\underline{x}_M)\, \nu(\underline{x}_{\bar{M}})$, which is equivalent to the following condition [7]: either, $\mathcal{H}_{M\bar{M}} = \emptyset$ or $\mathcal{K}_{M\bar{M}} = \emptyset$.

The next corollary is proposed without proof since it results immediately from theorems 4.1 and 4.2 (see Definition 3.5 and conventions 3.8).

Corollary 4.7: Let (C,Φ) be some binary (broad-sense) coherent system; a simple coherent decomposition of degree (d,d') $((d,d')\in \mathbf{N}^{*2})$ of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ can be defined from the partition $\{M,\bar{M}\}$ of C if and only if there exist some binary broad-sense coherent systems, (M,μ_r), $r\in D$, and $(\bar{M},\nu_r)$, $r \in D'$, which verify the following relations:

(i) $d = |\mathcal{J}(((M,\mu_r))_{r\in D})|$ and $d' = |\mathcal{J}(((\bar{M},\nu_r))_{r\in D'})|$;

(ii) let $D_2 = D \cap D'$, $D_1 = D \setminus D_2$ and $D_3 = D' \setminus D_2$; then, for every $\underline{x} \in S_C$,

$$\Phi(\underline{x}) = \Big(\coprod_{r\in D_1} \mu_r(\underline{x}_M)\Big) \vee \Big(\coprod_{r\in D_2} \mu_r(\underline{x}_M)\, \nu_r(\underline{x}_{\bar{M}})\Big) \vee \Big(\coprod_{r\in D_3} \nu_r(\underline{x}_{\bar{M}})\Big)$$

$$\Big(\Phi(\underline{x}) = \Big(\prod_{r\in D_1} \mu_r(\underline{x}_M)\Big).\Big(\prod_{r\in D_2} (\mu_r(\underline{x}_M) \vee \nu_r(\underline{x}_{\bar{M}}))\Big).\Big(\prod_{r\in D_3} \nu_r(\underline{x}_{\bar{M}})\Big)\Big).$$

5. SOME SET-CHARACTERIZATIONS; TWO STRONG CASES FOR THE PSEUDO-MODULARITY:

5.1. Some Tests for the Simple Coherent Subsystems and for the Pseudo-Modules:

Notations 5.1: Given two collections of subsets of C, $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \,\mathcal{U}\, \mathcal{B} = \{A \cup B \;/\; A \in \mathcal{A},\ B \in \mathcal{B}\}$.

The next theorem and its corollaries generalize the well-known "test for modularity" proposed in [3;theorem 4.3] and proved in set terms in [7]. It is stated with notations 2.13 and 4.4.

Theorem 5.2: ("Tests for the Simple Coherent Subsystems")
A non-empty subset M of C, relevant to some binary broad-sense coherent system, (C,Φ), is a simple coherent subset of degree d of (C,Φ) with respect to the collection of its minimal path (cut) sets if and

only if there exist some collection of path (cut) sets of (C,Φ), $\mathcal{A}$, and d non-empty sub-collections of $\mathcal{A}$, $\mathcal{A}_r$, $r = 1,\ldots,d$, which satisfy the following conditions: let $D = \{r\in\mathbf{N}^* \,/\, r\leq d\}$;

(5.1) $\mathcal{H}_M \subset \mathcal{A}$ $(\mathcal{K}_M \subset \mathcal{A})$ and $\mathcal{A} = \mathcal{A}_M = \bigcup_{r=1}^{d} \mathcal{A}_r$;

(5.2) for every $r\in D$, $\mathcal{A}_r$ satisfies the incomparableness conditions (2.3);

(5.3) for every $(r,s)\in D^2$, if $r \neq s$ then, $\mathfrak{I}_M(\mathcal{A}_r) \neq \mathfrak{I}_M(\mathcal{A}_s)$;

(5.4) for every $r\in D$ and for every $(A,A')\in \mathcal{A}_r^2$, $(A\cap M)\cup(A'\cap\bar{M})\in \mathcal{A}_r$.

Proof: First, that any simple coherent subset of degree d of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ satisfies the conditions (5.1) to (5.4) follows immediately from theorems 3.10, 4.1 and 4.2. Indeed, it suffices to consider the following collections (see (3.4)):

(5.5) $\mathcal{A} = \bigcup_{r=1}^{d} \mathcal{A}_r$ with

$$\forall r\in E,\ \mathcal{A}_r = \mathcal{H}(\mu_r)\,\mathcal{U}\,\mathcal{H}(\nu_r) \qquad (\mathcal{A}_r = \mathcal{K}(\mu_r)\,\mathcal{U}\,\mathcal{K}(\nu_r))$$
$$\forall r\in \bar{E},\ \mathcal{A}_r = \mathcal{H}(\mu_r) \qquad (\mathcal{A}_r = \mathcal{K}(\mu_r)).$$

where $E\subset D$ is defined as in remark 4.3 while the binary coherent systems $(\bar{M},\nu_r)$, $r\in E$, and $(\bar{M},\nu)$ have been defined in (4.9) from the concerned simple coherent subsystem, $\mathcal{M} = \{(M,\mu_r) \,/\, r\in D\}$.

Conversely, by theorems 4.1 and 4.2, it suffices to show that the conditions (5.1) to (5.4) can ensure the existence of some binary coherent systems which verify (4.8). By proposition 3.7, it suffices to check it for some collections of path sets of (C,Φ), for instance.

First, let $A\in \mathcal{A}_r$ for some $r\in D$; if $A\subset M$, then, by (5.4), for every $A'\in \mathcal{A}_r$, $(A'\cap M)\cup(A\cap\bar{M}) = A'\cap M\in \mathcal{A}_r$; consequently, by (5.2), A' must be included in M. So, there exists some subset E of D such that:

(5.6) $\mathcal{A}_{M\bar{M}} = \bigcup_{r\in E} \mathcal{A}_r$ and $\mathcal{A}_M \setminus \mathcal{A}_{M\bar{M}} = \bigcup_{r\in\bar{E}} \mathcal{A}_r$.

Second, (5.2) and (5.4) ensure the existence of some binary broad-sense coherent systems, (M,μ_r), $r\in D$, and $(\bar{M},\nu_r)$, $r\in E$, respectively defined as follows: $\mathcal{H}(\mu_r) = \mathfrak{I}_M(\mathcal{A}_r)$ and $\mathcal{H}(\nu_r) = \mathfrak{I}_{\bar{M}}(\mathcal{A}_r)$. To check it, it suffices to show that these collections satisfy the incomparableness conditions (2.3). According to (5.6), two cases must be distinguished:

- if $E \neq D$, for every $r\in\bar{E}$, $\mathfrak{I}_M(\mathcal{A}_r) = \mathcal{A}_r$ (see (5.2));
- if $E \neq \emptyset$, let $r\in E$ and assume there is some $(A,A')\in\mathcal{A}_r^2$ such that

$A\cap M \subsetneqq A'\cap M$; let $A'' = (A\cap M)\cup(A'\cap\bar{M})$; then, $A'' \subsetneqq A'$ but this is inconsistent with (5.2) since $A''\in\mathcal{A}_r$, by (5.4).

Consequently, the collections $\mathcal{J}_M(\mathcal{A}_r)$, $r\in D$, satisfy the incomparableness conditions (2.3); by theorem 2.12, this ensures the existence of the binary coherent systems (M,μ_r), $r\in D$, and all of them are different, by (5.3). The existence of the binary coherent systems $(\bar{M},\nu_r)$, $r\in E$, can be proved in an analogous way since M and $\bar{M}$ play a symmetrical part in (5.4).

Furthermore, if $\mathcal{H}_{\bar{M}} \neq \mathcal{H}_{M\bar{M}}$, the existence of the binary broad-sense coherent system, $(\bar{M},\nu)$, such that: $\mathcal{H}(\nu) = \mathcal{H}_{\bar{M}}\setminus\mathcal{H}_{M\bar{M}}$, is immediate (see Theorem 2.12). If $\mathcal{H}_{\bar{M}} = \mathcal{H}_{M\bar{M}}$, let $\nu \equiv 0$.

Now, the result follows immediately from the definition of the binary systems thus brought into play (see Remark 4.3): for every $\underline{x}\in S_C$, $\Phi(\underline{x}) = 1$

$$\Longleftrightarrow \begin{cases} \text{either } \exists H\in(\mathcal{H}_{\bar{M}}\setminus\mathcal{H}_{M\bar{M}}) : (\underline{1}_H,\underline{0}_{\bar{H}\cap\bar{M}}) \le \underline{x}_{\bar{M}} \\ \text{or } \exists H\in\mathcal{A}_r, \text{ for some } r\in D : (\underline{1}_H,\underline{0}_{\bar{H}}) \le \underline{x} \qquad \text{(by (5.1))} \end{cases}$$

$$\Longleftrightarrow \nu(\underline{x}_{\bar{M}}) \vee (\coprod_{r\in E} \mu_r(\underline{x}_M)\,\nu_r(\underline{x}_{\bar{M}})) \vee (\coprod_{r\in\bar{E}} \mu_r(\underline{x}_M)) = 1.$$

So, (4.8) holds and M is a simple coherent subset of degree d of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $\square$

It should be noted that the collection $\mathcal{A}$ and its covering $\{\mathcal{A}_r \,/\, r=1,\ldots,d\}$ thus have been implicitely shown to be uniquely defined by (5.5) from the concerned simple coherent subsystem (see Remark 4.3). So, by corollary 3.13 (see (4.9)), a simple coherent subsystem satisfies (3.5) if and only if $\mathcal{A} = \mathcal{H}_M$ ($\mathcal{A} = \mathcal{K}_M$). Some stronger result holds for the pseudo-modules. This is shown with the next result (stated with notations 4.4).

Corollary 5.3: ("Tests for Pseudo-Modularity")

With the same assumptions as in theorem 5.2, M is a pseudo-modular subset of degree d of (C,Φ) with respect to the collection of its minimal path (cut) sets if and only if there exists some partition $\{\mathcal{A}_r \,/\, r=1,\ldots,d\}$ of $\mathcal{H}_M$ ($\mathcal{K}_M$) which satisfies the conditions (5.3) and (5.4).

Proof: This result can be proved directly with some arguments analogous to the ones proposed for theorem 5.2. In particular, this approach allows to put in evidence that for the particular case of the pseudo-modules, (5.4) may be weakened as follows without altering the concerned result: for every $r = 1,\dots,d$ and for every $(A,A')\in \mathcal{A}_r^2$, $(A\cap M)\cup(A'\cap\bar{M})\in \mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$.

But, this corollary can be deduced, in a shorter way, from theorem 5.2: according to it, any pseudo-modular subset of degree d of (C,Φ) with respect to $\mathcal{H}(\Phi)$ $(\mathcal{K}(\Phi))$ can be (uniquely) defined by (5.5); by corollary 3.17, $\mathcal{A} = \mathcal{H}_M$ $(\mathcal{K}_M)$ (see (4.10)); and by (3.6), $\{\mathcal{A}_r \ / \ r=1,\dots,d\}$ must be a partition of $\mathcal{H}_M$ $(\mathcal{K}_M)$. This latest point can be easily checked by contraposition.

Conversely, it appears immediately that any partition of $\mathcal{H}_M$ $(\mathcal{K}_M)$ satisfies (5.1) and (5.2). So, if a partition $\{\mathcal{A}_r \ / \ r=1,\dots,d\}$ of $\mathcal{H}_M$ $(\mathcal{K}_M)$ satisfies (5.3) and (5.4), then, by theorem 5.2, the corresponding simple coherent subsystem (or decomposition) is uniquely defined by (5.5). Assume it does not satisfy (3.6): for instance, in terms of the minimal path sets, there exist $(r,s)\in D^2$, $(A,A')\in \mathcal{H}(\mu_r)\times\mathcal{H}(\mu_s)$ and $(B,B')\in \mathcal{H}(\kappa)^2$ such that $r\neq s$, $A\subset A'$ and $B\cap\bar{M}\subset B'\cap\bar{M}$. By (4.9), (5.4) and (5.5), $H = A\cup(B\cap\bar{M})\in \mathcal{A}_r$ and $H' = A'\cup(B'\cap\bar{M})\in \mathcal{A}_s$. Consequently, there exists some $(r,s)\in D^2$, $r\neq s$, such that either $\mathcal{A}_r\cap\mathcal{A}_s \neq \emptyset$ or $H \subsetneq H'$, for some $(H,H')\in \mathcal{A}_r\times\mathcal{A}_s$. Of course, both of these relations cannot hold since $\{\mathcal{A}_r \ / \ r\in D\}$ is a partition of $\mathcal{H}_M\subset\mathcal{H}(\Phi)$. Consequently, (3.6) must hold. By proposition 3.16, the corresponding result with respect to $\mathcal{K}(\Phi)$ follows immediately by duality □

Remark 5.4: The tests for pseudo-modularity are always verified and consequently, useless on $(\mathcal{H}_M\setminus\mathcal{H}_{M\bar{M}})$ $((\mathcal{K}_M\setminus\mathcal{K}_{M\bar{M}}))$. But, some relations of equivalence appear naturally helpful for determining a pseudo-module on $\mathcal{H}_{M\bar{M}}$ $(\mathcal{K}_{M\bar{M}})$. This is shown hereafter.

Notations 5.5: Let $\mathcal{A}$ be some collection of non-empty subsets of C and B a non-empty subset of C such that for every $A\in\mathcal{A}$, $A\cap B\neq\emptyset$; let $\equiv_B$ be the equivalence relation defined on $\mathcal{A}$ as follows: for every $(A,A')\in\mathcal{A}^2$, $A \equiv_B A'$ if and only if $A\cap B = A'\cap B$.

Let $\approx_B$ be the equivalence relation defined as follows: for every $\mathcal{E}\subset\mathcal{A}$ and $\mathcal{F}\subset\mathcal{A}$, $\mathcal{E}\approx_B\mathcal{F}$ if and only if $\mathcal{J}_B(\mathcal{E}) = \mathcal{J}_B(\mathcal{F})$ (see Notations 2.13).

For any equivalence relation $\mathcal{R}$ defined on some collection $\mathcal{A}$, the quotient-set of $\mathcal{A}$ modulo $\mathcal{R}$ is designated as $\mathcal{A}/\mathcal{R}$. In addition,

(5.7) For $A = M$ or $\bar{M}$, let $\{\mathcal{B}_v \,/\, v \in B\} = \mathcal{H}_{M\bar{M}}/\equiv_{\bar{A}}$ $(\mathcal{K}_{M\bar{M}}/\equiv_{\bar{A}})$ and $\{\mathcal{E}_u / u \in E\} = [\mathcal{H}_{M\bar{M}}/\equiv_{\bar{A}}]/\approx_A$ $([\mathcal{K}_{M\bar{M}}/\equiv_{\bar{A}}]/\approx_A)$; then, $\mathcal{A}(A) = \{\mathcal{A}_u = \bigcup_{\mathcal{B} \in \mathcal{E}_u} \mathcal{B} \,/\, u \in E\}$ is a partition of $\mathcal{H}_{M\bar{M}}$ $(\mathcal{K}_{M\bar{M}})$.

Proposition 5.6: Given some binary broad-sense coherent system, (C,Φ), let M be some non-empty subset of C and assume $\mathcal{H}_{M\bar{M}} \neq \emptyset$ $(\mathcal{K}_{M\bar{M}} \neq \emptyset)$; then, some pseudo-module of (C,Φ) with respect to the collection of its minimal path (cut) sets can always be defined on $\mathcal{H}_{M\bar{M}}$ $(\mathcal{K}_{M\bar{M}})$ with the partition $\mathcal{A}(M)$ or $\mathcal{A}(\bar{M})$ defined by (5.7).

Proof: Of course, by proposition 3.16, it suffices to prove this result with respect to $\mathcal{H}(\Phi)$, for instance. In addition, the parts played by M and $\bar{M}$ are quite symmetrical since $\mathcal{H}_{M\bar{M}}$ is assumed non-empty and it suffices to consider the case $A = M$, for instance. Indeed, the result can be deduced, in the shortest way, from corollary 5.3 and the main part of the proof consists in showing that the partition $\mathcal{A}(M)$ satisfies (5.4). First, for every $u \in E$ and every $(A,A') \in \mathcal{A}_u^2$, two cases can be distinguished:

case 1: $(A,A') \in \mathcal{B}_v^2$, for some $v \in B$ such that $\mathcal{B}_v \in \mathcal{E}_u$; then, $A \cap \bar{M} = A' \cap \bar{M}$ and therefore, $(A \cap M) \cup (A' \cap \bar{M}) = A \in \mathcal{B}_v$; now, $\mathcal{B}_v \subset \mathcal{A}_u$ and consequently, $(A \cap M) \cup (A' \cap \bar{M}) \in \mathcal{A}_u$;

case 2: $(A,A') \in \mathcal{B}_v \times \mathcal{B}_w$, for some $(v,w) \in B^2$ such that $v \neq w$ and $(\mathcal{B}_v, \mathcal{B}_w) \in \mathcal{E}_u^2$; so, $\mathcal{B}_v \approx_M \mathcal{B}_w$ (see Notations 5.5) and therefore, for instance, $A \cap M = A_0 \cap M$, for some $A_0 \in \mathcal{B}_w$; consequently (see case 1), $(A \cap M) \cup (A' \cap \bar{M}) = (A_0 \cap M) \cup (A' \cap \bar{M}) \in \mathcal{B}_w$; now, $\mathcal{B}_w \subset \mathcal{A}_u$ and consequently, $(A \cap M) \cup (A' \cap \bar{M}) \in \mathcal{A}_u$.

The result now is immediate (see Remark 5.4) since for every $(u,v) \in E^2$, $u \neq v$, $\mathcal{I}_M(\mathcal{A}_u) \neq \mathcal{I}_M(\mathcal{A}_v)$ by definition of $\approx_M$ $\square$

Remark 5.7: The pseudo-modules of a binary coherent system, (C,Φ), can be easily determined by means of the equivalence relations defined in notations 5.5. This can be shown in terms of $\mathcal{A}(M)$ (see (5.7)) and of the minimal path sets, for instance.

Let $\mathcal{H}_{M\bar{M}} = \{H_r \,/\, r \in I\}$, for some $I \subset \{r \in \mathbf{N}^* \,/\, r \leq h\}$ (see remark 2.11). With the same conventions as in (5.7), let $\{B_v \,/\, v \in B\}$ be the partition of I corresponding to the partition $\mathcal{H}_{M\bar{M}}/\equiv_{\bar{M}}$ of $\mathcal{H}_{M\bar{M}}$: for every $v\in B$, $\mathcal{B}_v = \{H_r \in \mathcal{H}_{M\bar{M}} \ / \ r\in B_v\}$ and for each $v\in B$, choose some $r(v)\in B_v$. With the same conventions as in remark 4.5, for every $\underline{x}\in S_C$,

$$\Phi(\underline{x}) = \mu(\underline{x}_M) \vee \Big(\coprod_{v\in B} \big(\coprod_{r\in B_v} \eta_{r/M}(\underline{x}_M)\big)\eta_{r(v)/\bar{M}}(\underline{x}_{\bar{M}})\Big) \vee \nu(\underline{x}_{\bar{M}}),$$

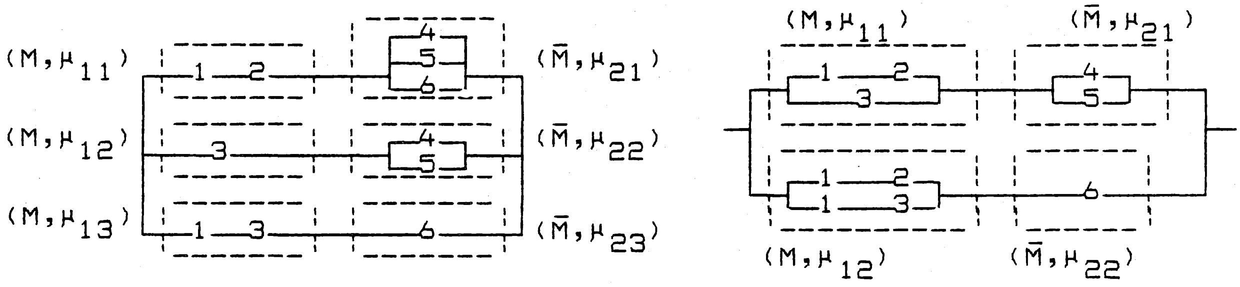
where $\mu \equiv \coprod_{r\in E_M} \eta_{r/M}$ and $\nu \equiv \coprod_{r\in E_{\bar{M}}} \eta_{r/\bar{M}}$.

Let $\{E_u \ / \ u\in E\}$ be the partition of B corresponding to $[\mathcal{H}_{M\bar{M}}/\equiv_{\bar{M}}]/\approx_M$ (see (5.7)): for every $u\in E$, $\mathcal{E}_u = \{\mathcal{B}_v \in [\mathcal{H}_{M\bar{M}}/\equiv_{\bar{M}}] \ / \ v\in E_u\}$ and for each $u\in E$, choose some $v(u)\in E_u$. With these conventions, for every $u\in E$ and for every $v\in E_u$, $\coprod_{r\in B_v} \eta_{r/M} \equiv \coprod_{r\in B_{v(u)}} \eta_{r/M}$.

So, a pseudo-modular decomposition of (C,Φ) with respect to $\mathcal{H}(\Phi)$ can be immediately defined from M (see Remark 4.3): for every $\underline{x}\in S_C$,

$$\begin{aligned}\Phi(\underline{x}) &= \mu(\underline{x}_M) \vee \Big(\coprod_{u\in E} \ \coprod_{v\in E_u} \big(\coprod_{r\in B_v} \eta_{r/M}(\underline{x}_M)\big).\eta_{r(v)/\bar{M}}(\underline{x}_{\bar{M}})\Big) \vee \nu(\underline{x}_{\bar{M}}) \\ &= \mu(\underline{x}_M) \vee \Big(\coprod_{u\in E} \big(\coprod_{r\in B_{v(u)}} \eta_{r/M}(\underline{x}_M)\big).\big(\coprod_{v\in E_u} \eta_{r(v)/\bar{M}}(\underline{x}_{\bar{M}})\big)\Big) \vee \nu(\underline{x}_{\bar{M}})\end{aligned}$$

Indeed, by theorem 4.1, this yields an alternative proof for proposition 5.6. Of course, a pseudo-modular decomposition of (C,Φ) with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) can be determined from $\bar{M}$ with the same procedure, by means of the equivalence relations $\equiv_M$ and $\approx_{\bar{M}}$. Both of the pseudo-modular decompositions thus obtained may be different (see Figure 3) but they are tightly related: as an indication, for every $\mathcal{A}\in\mathcal{H}_{M\bar{M}}/\equiv_{\bar{M}}$ and every $\mathcal{B}\in\mathcal{H}_{M\bar{M}}/\equiv_{\bar{M}}$, $\mathcal{A}\approx_M\mathcal{B}$ if and only if $|\mathcal{A}| = |\mathcal{B}|$ and for every $A\in\mathcal{A}$, there exists some $B\in\mathcal{B}$ such that $A \equiv_M B$.



$C = \{i\in \mathbf{N}^* \ / \ i\leq 6\}$ and $M = \{i\in C \ / \ i\leq 3\}$.

Figure 3: Pseudo-Modular Decompositions of a Binary Coherent System, (C,Φ), with respect to $\mathcal{H}(\Phi)$ and respectively determined with $\equiv_M$ and $\approx_{\bar{M}}$ (left) and $\equiv_{\bar{M}}$ and $\approx_M$ (right).

In addition, if $\mathcal{H}(\Phi) = \mathcal{H}_{M\bar{M}}$ ($\mathcal{K}(\Phi) = \mathcal{K}_{M\bar{M}}$), a modular decomposition of (C,Φ) can be defined from the partition $\{M,\bar{M}\}$ of C if and only if the rough partition is induced by $\equiv_M$, $\approx_M$, $\equiv_{\bar{M}}$ or $\approx_{\bar{M}}$ on $\mathcal{H}_{M\bar{M}}$, $\mathcal{H}_{M\bar{M}}/\equiv_{\bar{M}}$, $\mathcal{H}_{M\bar{M}}$ or $\mathcal{H}_{M\bar{M}}/\equiv_M$ ($\mathcal{K}_{M\bar{M}}$, $\mathcal{K}_{M\bar{M}}/\equiv_{\bar{M}}$, $\mathcal{K}_{M\bar{M}}$ or $\mathcal{K}_{M\bar{M}}/\equiv_M$), respectively.

5.2. Strong Pseudo-Modularity:

Remark 5.8: By (5.5), a trivial pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) is defined from M (see Remark 4.5) if and only if the corresponding partition $\{ \mathcal{A}_r$ / r$\in$D$\}$ of $\mathcal{H}_M$ ($\mathcal{K}_M$) (see Corollary 5.3) verifies the following equality: $\{ \mathfrak{I}_M(\mathcal{A}_r)$ / r$\in$D$\}$ is the discrete partition of $\mathfrak{I}_M(\mathcal{H}_M)$ ($\mathfrak{I}_M(\mathcal{K}_M)$). So, if $|\mathcal{H}_M \setminus \mathcal{H}_{M\bar{M}}| > 1$ ($|\mathcal{K}_M \setminus \mathcal{K}_{M\bar{M}}| > 1$), a non-trivial pseudo-module of (C,Φ) with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) can always be defined from M (see Remark 5.4). However, in an intuitive way, "complex systems" give raise to a "large number of repeated components" in their minimal path sets and cut sets. So, the most efficient pseudo-modular decompositions of (C,Φ) are defined from some pseudo-modular subset M which can yield some non-trivial decomposition for the system part corresponding to $\mathcal{H}_{M\bar{M}}$ ($\mathcal{K}_{M\bar{M}}$). Such pseudo-modular subsets are introduced hereafter.

Definition 5.9: Let M be a proper subset of C, relevant to some binary coherent system, (C,Φ), and such that $|M| > 1$.
Let $\mathcal{M} = \{(M,\mu_r)$ / r$\in$D$\}$ be a pseudo-module of (C,Φ), with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$); let E$\subset$D be defined from $\mathcal{M}$ as in remark 4.3.
$\mathcal{M}$ is a **strong pseudo-module** of (C,Φ) if and only if:

(5.8) either $E = \emptyset$;

(5.9) or $\{(M,\mu_r)$ / r$\in$E$\}$ is different from $\{(M,\eta_{r/M})$ / r$\in D_{M\cap\bar{M}}\}$ ($\{(M,\theta_{r/M})$ / r$\in D_{M\cap\bar{M}}\}$), where $D_{M\cap\bar{M}}$ has been defined with the trivial pseudo-module $\{(M,\eta_{r/M})$ / r$\in D_M\}$ ($\{(M,\theta_{r/M})$ / r$\in D_M\}$) in remark 4.5.

Then, M is a **strong pseudo-modular subset** of (C,Φ).

Definition 5.10: A **strong pseudo-modular decomposition** of (C,Φ) is defined with some strong pseudo-modules of (C,Φ), only.

The next result follows immediately from corollary 5.3 (see

Remark 5.8).

Corollary 5.11: Let M be specified as in definition 5.9; M is a strong pseudo-modular subset of (C,Φ) with respect to $\mathscr{H}(\Phi)$ $(\mathfrak{H}(\Phi))$ if and only if

(5.10) either $\mathscr{H}_{M\bar{M}} = \emptyset$ $(\mathfrak{H}_{M\bar{M}} = \emptyset)$;

(5.11) or there exists some partition $\{\mathcal{A}_r \,/\, r\in E\}$ of $\mathscr{H}_{M\bar{M}}$ $(\mathfrak{H}_{M\bar{M}})$ which satisfies (5.3) and (5.4) as also the following condition: the collection $\{\mathfrak{J}_M(\mathcal{A}_r) \,/\, r\in E\}$ is different from the discrete partition of $\mathfrak{J}_M(\mathscr{H}_{M\bar{M}})$ $(\mathfrak{J}_M(\mathfrak{H}_{M\bar{M}}))$.

Remark 5.12: By remark 4.6, (5.8) or (5.10) is a necessary and sufficient condition for ensuring the existence of some modular decomposition defined from the partition $\{M,\bar{M}\}$ of C. Indeed, any non-trivial module of (C,Φ), (M,μ), is a strong pseudo-module of degree 1 of (C,Φ): it corresponds to the rough partition of $\mathscr{H}_M$ and of $\mathfrak{H}_M$. In particular, if $\mathscr{H}_{M\bar{M}} = \emptyset$ $(\mathfrak{H}_{M\bar{M}} = \emptyset)$, then, $\mathfrak{H}_M = \mathfrak{H}_{M\bar{M}}$ $(\mathscr{H}_M = \mathscr{H}_{M\bar{M}})$.

Examples 5.13: It can be easily checked that $\mathcal{M}_2$, defined in examples 3.24, is a strong pseudo-modular decomposition for the 2-out-of-4 system both with respect to $\mathscr{H}(\Phi)$ and $\mathfrak{H}(\Phi)$. Another strong pseudo-modular decomposition has been given in figure 3 (right).

In addition, let (C,Φ) be a binary strict-sense coherent system of order 8 defined as follows: $C = \{i\in \mathbb{N}^* \,/\, i\leq 8\}$ and for every $\underline{x}\in S^8$,

$\Phi(\underline{x}) = \coprod_{j=1}^{2} \mu_{1j}(\underline{x}_M).\mu_{2j}(\underline{x}_{\bar{M}})$, with $M = \{i\in \mathbf{N}^* \,/\, i\leq 4\}$

$\mathscr{H}(\mu_{11}) = \{\{1,2\};\{3,4\}\}$, $\mathscr{H}(\mu_{12}) = \{\{1\};\{4\}\}$, $\mathscr{H}(\mu_{21}) = \{\{5\};\{7\}\}$ and $\mathscr{H}(\mu_{22}) = \{\{5,6\};\{7,8\}\}$.

Indeed, $\mathscr{H}(\Phi) = \mathscr{H}_{M\bar{M}}$ and (C,Φ) is thus defined by means of a strong pseudo-modular decomposition of degree (2,2) with respect to $\mathscr{H}(\Phi)$ (see (5.5) and Corollary 5.11).

The next result follows immediately from proposition 5.6 and corollary 5.11 (see Remark 5.7).

Proposition 5.14: Let M be specified as in definition 5.5; if either (5.10) holds or the discrete partition is not induced by the equivalence relation $\equiv_{\bar{M}}$ or $\approx_{\bar{M}}$ on $\mathscr{H}_{M\bar{M}}$ or $\mathscr{H}_{M\bar{M}}/\equiv_M$ $(\mathfrak{H}_{M\bar{M}}$ or $\mathfrak{H}_{M\bar{M}}/\equiv_M)$, respectively, then, M is a strong pseudo-modular subset of (C,Φ) with respect to $\mathcal{P}(\Phi)$ $(\mathcal{K}(\Phi))$.

5.3. Quasi-Modularity:

The special type of pseudo-modularity introduced hereafter intuitively seems to be tightly related to the strong pseudo-modularity.

Definition 5.15: $\mathcal{M} = \{(M,\mu_r) \ / \ r=1,\ldots,d\}$ is a **quasi-module of degree d** of (C,Φ) if and only if $\mathcal{M}$ is a pseudo-module of degree d of (C,Φ) both with respect to $\mathcal{P}(\Phi)$ and $\mathcal{K}(\Phi)$.
Then, M is a **quasi-modular subset of degree d** of (C,Φ).

Definition 5.16: A **quasi-modular decomposition** of (C,Φ) is a coherent decomposition of (C,Φ) all the elements of which are some quasi-modules of (C,Φ).

Examples 5.17: The pseudo-modular decomposition $\mathcal{M}_2$ which has been defined in examples 2.24 is a quasi-modular decomposition for the 2-out-of-4 system. Of course, any module of (C,Φ) is a quasi-module of degree 1 of (C,Φ).

The next result follows immediately from corollary 5.3 and constitutes some straight extension of the set-characterizations proposed in [7] for modules (see Remark 2.14).

Corollary 5.18: ("Test for the Quasi-modularity")
With the same assumptions as in theorem 5.2, M is a quasi-module of degree d of (C,Φ) if and only if there exist some partitions of $\mathcal{P}_M$ and of $\mathcal{K}_M$, $\mathcal{A} = \{\mathcal{A}_r \ / \ r=1,\ldots,d\}$ and $\mathcal{B} = \{\mathcal{B}_r \ / \ r=1,\ldots,d\}$ respectively, which satisfy the following conditions:

(a) $\mathcal{A}$ and $\mathcal{B}$ satisfy (5.4) while $\mathcal{A}$ or $\mathcal{B}$ satisfies (5.3);

(b) for each $r = 1,\ldots,d$, $\mathcal{I}_M(\mathcal{A}_r)$ and $\mathcal{I}_M(\mathcal{B}_r)$ gather together the minimal path sets and the minimal cut sets, respectively, of some binary coherent system, (M,μ_r).

Remark 5.19: An extension of "the three modules theorem" [3][7] in terms of pseudo-modules cannot be considered since any non-empty subset of C relevant to (C,Φ) can be a pseudo-modular subset of (C,Φ) (see Remark 4.5). But, such an extension could be investigated in terms of the strong pseudo-modularity and in terms of the quasi-modularity and could yield some new criteria for determining the strong pseudo-modules, the quasi-modules and the modules of a binary coherent system, by means of the elementary set-operations. However, this is out of the scope of this study.

6. MONOTONE PSEUDO-MODULES AND PERFORMANCE FUNCTIONS:

6.1. An Extension of the Application Domain of the Performance Functions:

The availability of a binary coherent system can be determined from the availabilities of its modules, then assumed mutually independent, by means of the performance function of the organizing system of the concerned modular decomposition [3]. This result can be extended in terms of some pseudo-modular decompositions introduced in this section with the monotone pseudo-modules. First, some extension of the application domain currently considered for the performance function [2] must be put in evidence.

Notations 6.1: Any permutation π, defined on some finite set I, will be designated as the $|I|$-uple of its images, $(i_r)_{r\in I}$: for every $r\in I$, $i_r = \pi(r)$.

Notations 6.2: A binary broad-sense coherent system, (C,Φ), is assumed to be observed at an arbitrarily fixed point in time and for every $i\in C$ respectively, the random variable X_i gives an account of the performance level of the component i; let $\underline{X} = (X_i)_{i\in C}$; then, $\Phi(\underline{X})$ gives an account of the performance level of (C,Φ) at the concerned point in time.

The **performance function** [12][2] of (C,Φ) allows to determine the availability of (C,Φ) in terms of the availabilities of its components if their performance variables, X_i, $i\in C$, are mutually independent: $h_\Phi(\underline{p}) = P[\Phi(\underline{X})=1]$, where $\underline{p} = (p_i)_{i\in C}\in[0,1]^{|C|}$ and for every $i\in C$, $p_i = P[X_i=1]$.

This "classical" result can be extended with the next theorem. First, some conventions stated in notations 2.1 must be extended to $[0,1]^{|C|}$: as an indication, given a partition $\{M_r \;/\; r=1,\ldots,m\}$ of C, any vector $\underline{u}\in[0,1]^{|C|}$ can be defined as follows:
$\underline{u} = (\underline{u}_{M_r})_r \in \overset{m}{\underset{r=1}{\times}} [0,1]^{|M_r|}$, where $\underline{u}_{M_r} = (\underline{u}_i)_{i\in M_r}$, $r = 1, \ldots, m$.

Theorem 6.3: Given a binary broad-sense coherent system, (C,Φ), if (its components are submitted to "some constraints" such that) there exists some partition of C, $\{C_j \;/\; j=1,\ldots,c\}$ for some $c\in\mathbf{N}^*$, which satisfies both of the following conditions:

(6.1) for each $j=1,\ldots,c$, such that $|C_j|>1$, the performance variables of the components in C_j, X_i, $i\in C_j$, can be fully ordered: there exists some permutation on C_j, $(i_s)_{s\in C_j}$, such that for every $(r,s)\in C_j^2$, if $i_r \le i_s$, then, $X_{i_r} \overset{as}{\le} X_{i_s}$,

(6.2) the performance vectors, $(X_i)_{i\in C_j}$, $j=1,\ldots,c$, are mutually independent,

then, the availability of (C,Φ) can be determined by means of its performance function: for every $\underline{p}\in[0,1]^{|C|}$, $h_\Phi(\underline{p}) = P[\Phi(\underline{X})=1]$. Moreover, the performance function h_Φ is multilinear and non-decreasing in each of its arguments from $[0,1]^{|C|}$ into $[0,1]$. In addition, for every $\underline{p}\in[0,1]^{|C|}$, $h_{\Phi^D}(\underline{p}) = 1 - h_\Phi(\underline{1}-\underline{p})$.

Proof: First, the existence of the performance function of (C,Φ) can be put in evidence with the Poincaré-Sylvester formula:

$$h_\Phi(\underline{p}) = P[\Phi(\underline{X})=1] = P[\bigcup_{r=1}^{h} [\bigcap_{i\in H_r} [X_i=1]]]$$
$$= \sum_{s=1}^{h} (-1)^{s-1} \sum_{r_1=1}^{h} \cdots \sum_{r_s=r_{s-1}+1}^{h} P[\bigcap_{u=1}^{s} \bigcap_{i\in H_{r_u}} [X_i=1]]$$

For every $(r_1,\ldots,r_s)$ in the sum above, let
$J(r_1,\ldots,r_s) = \{j\in\{1,\ldots,c\} \;/\; \exists i\in C_j : i\in \bigcup_{u=1}^{s} H_{r_u}\}$
for every $j\in J(r_1,\ldots,r_s)$, let $I_j(r_1,\ldots,r_s) = C_j\cap(\bigcup_{u=1}^{s} H_{r_u})$ and
$X_{i_j(r_1,\ldots,r_s)} \overset{as}{=} \operatorname{Inf}\{X_i \;/\; i\in I_j(r_1,\ldots,r_s)\}$ (which exists by (6.1)).
With these conventions, by (6.1),

$$P[\bigcap_{u=1}^{s} \bigcap_{i\in H_{r_u}} [X_i=1]] = P[\bigcap_{j\in J(r_1,\ldots,r_s)} \bigcap_{i\in I_j(r_1,\ldots,r_s)} [X_i=1]] =$$
$$\prod_{j\in J(r_1,\ldots,r_s)} P[X_{i_j(r_1,\ldots,r_s)}=1] \quad \text{(by (6.2))}.$$

Now, it appears immediately that h_Φ is a multilinear function from

$[0,1]^{|C|}$ into $[0,1]$, the partial derivatives of which verify the following relation: for every $j \in \{1,\ldots,c\}$ and every $i \in C_j$,

$\delta h_\Phi(\underline{p})/\delta p_i = \delta h_\Phi(p_i, \underline{0}_{C_j \setminus \{i\}}, \underline{p}_{C \setminus C_j})/\delta p_i =$

$\delta(E[X_i \Phi(1_i, \underline{0}_{C_j \setminus \{i\}}, \underline{X}_{C \setminus C_j}) + (1-X_i)\Phi(\underline{0}_{C_j}, \underline{X}_{C \setminus C_j})])/\delta p_i =$

$E[\Phi(1_i, \underline{0}_{C_j \setminus \{i\}}, \underline{X}_{C \setminus C_j}) - \Phi(\underline{0}_{C_j}, \underline{X}_{C \setminus C_j})]$.

So, the non-decreasing property of Φ ensures the same property for h_Φ. In addition, the relation of duality is immediate □

6.2. Monotone Pseudo-Modules:

The monotone pseudo-modules can be easily introduced by means of the dominance relation [18], briefly reviewed hereafter.

Definition 6.4: Given two binary broad-sense coherent systems, (C,Φ) and (C,Γ), the latter one **dominates** the former one if and only if for every $\underline{x} \in S_C$, $\Phi(\underline{x}) \leq \Gamma(\underline{x})$.

Proposition 6.5: (C,Γ) dominates (C,Φ) if and only if (C,Φ^D) dominates (C,Γ^D).

Proposition 6.6: (C,Γ) dominates (C,Φ) if and only if one of the following (equivalent) conditions holds:

(i) for every $H \in \mathcal{H}(\Phi)$, there exists some $H' \in \mathcal{H}(\Gamma)$ such that $H' \subset H$;

(ii) for every $K' \in \mathcal{K}(\Gamma)$, there exists some $K \in \mathcal{K}(\Phi)$ such that $K \subset K'$.

Definition 6.7: Given an arbitrary broad-sense coherent system, (C,Φ), a pseudo-module $\{(M,\mu_r) \ / \ r=1,\ldots,d\}$ of (C,Φ) is said to be **monotone** if and only if there exists some permutation, $\pi = (i_r)_{r=1,\ldots,d}$, such that for every $r \in \{s \in \mathbf{N}^* \ / \ s \leq d\}$, $(M,\mu_{i_{r+1}})$ dominates (M,μ_{i_r}).

Then, $\{(M,\mu_r) \ / \ r=1,\ldots,d\}$ is a **monotone pseudo-module according to the permutation $(i_r)_{r=1,\ldots,d}$**.

A **monotone pseudo-modular decomposition** is defined with monotone pseudo-modules only.

Remark 6.8: The set-characterization of the monotone pseudo-modules can be immediately deduced from proposition 6.6 and the monotone pseudo-modularity appears immediately to be tightly related to the strong pseudo-modularity.

Of course, any module is a monotone pseudo-module. By proposition 6.6, the pseudo-modular decomposition defined in examples 5.13 as also the quasi-modular decomposition $\mathcal{M}_2$ defined in examples 2.24 (see Examples 5.17) are monotone.

The next result follows immediately from proposition 6.5 (see also Proposition 3.16).

Proposition 6.9: Given an arbitrary binary coherent system, (C,Φ),

- a pseudo-module $\{(M,\mu_r) \ / \ r=1,...,d\}$ of (C,Φ) is monotone if and only if the pseudo-module $\{(M,\mu_r^D) \ / \ r=1,...,d\}$ of (C,Φ^D) is itself monotone;
- a pseudo-modular decomposition $\{\{(M,\mu_{rv_r})/v_r=1,...,d_r\}/r=1,..,m\}$ of (C,Φ) is monotone if and only if the pseudo-modular decomposition $\{\{(M,\mu^D_{rv_r}) \ / \ v_r=1,...,d_r\} \ / \ r=1,..,m\}$ of (C,Φ^D) is itself monotone.

The next corollary follows immediately from theorem 6.3.

Corollary 6.10: Let $\mathcal{M} = \{\{(M_r,\mu_{rv_r}) \ / \ v_r=1,...,d_r\}/ \ r=1,...,m\}$ be a pseudo-modular decomposition of some binary broad-sense coherent system, (C,Φ), with respect to $\mathcal{H}(\Phi)$ ($\mathfrak{K}(\Phi)$).

If $\mathcal{M}$ is monotone and if the performance vectors of its pseudo-modules, $(\mu_{rv_r}(\underline{X}_{M_r}))_{v_r}$, $r = 1, ...,m$, are mutually independent, then, the availability of (C,Φ), $H_\Phi[\mathcal{L}(\underline{X})]$, can be determined from those of its pseudo-modules, $H_{\mu_{rv_r}}[\mathcal{L}(\underline{X}_{M_r})]$, $v_r = 1,...,d_r$, $r = 1,...,m$, by means of the performance function of the corresponding organizing system, $(C' = \bigcup_{r=1}^{d} \{r\}\times\{j\in\mathbf{N}^* \ / \ j\le d_r\},\boldsymbol{\kappa})$:

$$H_\Phi[\mathcal{L}(\underline{X})] = h_{\boldsymbol{\kappa}}((H_{\mu_{rv_r}}[\mathcal{L}(\underline{X}_{M_r})])_{(r,v_r)\in C'}).$$

In addition, if all the components of (C,Φ) are mutually independent, for every $\underline{p}\in[0,1]^{|C|}$, $h_\Phi(\underline{p}) = h_{\boldsymbol{\kappa}}((h_{\mu_{rv_r}}(\underline{p}_{M_r}))_{(r,v_r)\in C'})$.

7. SOME CONCLUSIONS:

Imagine that a complex binary coherent system has been designed by several teams, as it happens most often in reality. Assume all the subsystems which have been thus designed separately are coherent and the performance level of the complete system can be fully determined from their own performance levels, by means of some organizing system, itself coherent. By definition 3.3, the concerned system is thus defined by means of one of its coherent decompositions. Determine the minimal path sets or cut sets of the corresponding organizing system. By proposition 3.9, some simple coherent decomposition then can be easily deduced and by theorem 3.10 (see (3.4)), the minimal path sets or cut sets of the complete system can be easily determined. In an alternative way, by proposition 3.19, a pseudo-modular decomposition can be equally identified and by corollary 3.18, the minimal path sets or cut sets of the complete system can be determined in a simpler way or its main characteristics of **reliability** can be calculated in an "approximated way": this is shown in the next chapter. By corollary 6.10, some particular case allows to determine the reliability or the availability of the complete system in terms of those of its pseudo-modules, in an "exact" way.

In addition, as and when a complex system is designed, various solutions may be considered and the knowledge of a coherent decomposition may appear helpful for avoiding the complete analysis for all the systems then to be studied: two systems may differ in some of their coherent subsystems or in the organizing system of their coherent decompositions.

* *
*

Chapter B:

PSEUDO-MODULAR DECOMPOSITIONS AND "REFINED BOUNDS" FOR THE INTERVAL RELIABILITY AND FOR THE AVAILABILITY FOR BINARY COHERENT SYSTEMS

1. INTRODUCTION AND SUMMARY:

Two well-known papers are at the origin of the "refined bounds" currently proposed for the main characteristics of **reliability** for binary coherent systems: modules and modular decompositions [3] have been introduced in order to study a binary coherent system as some collection of smaller order systems themselves coherent (see Chapter A) while association has been introduced by Esary, Proschan and Walkup [14] with various characterizations; in particular, association has been shown to be sufficient for ensuring some basic inequalities previously considered in [12] for obtaining some "bounds" under the mutual independence. These "bounds" concern the availability (at some fixed point in time and consequently the reliability of "non-maintained systems"). Modular decompositions have been used by Bodin [5] for refining these "bounds" under the mutual independence while association has been considered by Esary and Proschan [13] for extending the same "bounds" to the reliability of "maintained systems". Association has been equally considered by Barlow and Proschan [2] for introducing some "simple bounds" for the availability. With various conditions of dependence ranging from association to the mutual independence, all the "bounds" obtained till then have been generalized for the interval (un-)reliability [1] (or interval (un-)availability [20]) and refined in terms of modular decompositions by Natvig [20].

Various results proposed in chapter A allow to appraise the great generality of the pseudo-modules. However, all the "refined bounds" obtained till now [5][20] can be generalized in terms of pseudo-modular decompositions. This is shown throughout this chapter.
First, the "basic bounds" [20] are reviewed in section 2, by following the order in which they can be proved in the shortest way [18] and by weakening the conditions of dependence under which some of them have been considered originally.
In section 3, all the "bounds" which have been proposed in [20] for the interval (un-)reliability are extended in terms of pseudo-modular decompositions. The particular case of the (instantaneous) availability is examined throughout section 4 while some comparisons are proposed in section 5, thus extending all the results proposed till now for "bounding" the main characteristics of **reliability** for "complex" binary coherent systems.
Some conclusions are proposed in section 6.

All the results proposed in this chapter concern the binary coherent systems, only. However, they should be retained as some fundamental results for obtaining the "refined bounds" for the multinary coherent systems, with some simple approach proposed in the second part of this study (see Preamble).

2. THREE TYPES OF "BASIC BOUNDS":

Notations 2.1: An arbitrary broad-sense coherent system, (C,Φ), is always considered with the conventions stated in chapter A.

The reference set of the time parameter is defined as some subset τ of $\mathbf{R}^+$ and as in [20], it can be discrete or continuous.

For every non-empty subset M of C, the S_M-valued stochastic process, $\underline{X}_{\mathbf{M}} = \{\underline{X}_M(t) \overset{as}{=} (X_i(t))_{i\in M} \ / \ t\in\tau\}$ (where "as" is an abbreviated form for almost-surely) is the **joint performance process** of the components in M; in particular, for every $i\in C$ respectively, $\mathbf{X_i} = \{X_i(t) \ / \ t\in\tau\}$ is the **marginal performance process** of the component i while $\underline{X} = \{\underline{X}(t) \overset{as}{=} (X_i(t))_{i\in C} \ / \ t\in\tau\}$ is the joint performance process of all the components of (C,Φ).

The random behaviour of (C,Φ) is assumed fully described by its performance process, $\{\Phi(\underline{X}(t)) \ / \ t\in\tau\}$ (see Remark A.2.11):

$$(2.1) \quad \forall t\in\tau, \ \Phi(\underline{X}(t)) \overset{as}{=} \coprod_{r=1}^{h} \eta_r(\underline{X}(t)) \overset{as}{=} \prod_{r=1}^{k} \theta_r(\underline{X}(t)).$$

All the stochastic processes to be considered are assumed to be right-continuous on τ. In particular, this is true for all the marginal performance processes $\mathbf{X_i}$, $i\in C$, and consequently, for all the performance processes yet defined or to be defined from them.

Notations 2.2: For any closed interval I in $\mathbf{R}^+$, the corresponding interval of time in τ is designated as $\tau(I)$: $\tau(I) = \tau\cap I$.

Let $\underline{p}(I) = (p_i(I))_{i\in C}\in[0,1]^{|C|}$ ($\underline{q}(I) = (q_i(I))_{i\in C}\in[0,1]^{|C|}$) be the vector of the components (un-)reliabilities in the time interval $\tau(I)$: for every $i\in C$ respectively, $p_i(I) = P[X_i(t)=1; \forall t\in\tau(I)]$ ($q_i(I) = P[X_i(t)=0; \forall t\in\tau(I)]$).

Let $\mathcal{L}(Y(I))$ be the distribution of an arbitrary stochastic process $\{Y(t) \ / \ t\in\tau\}$ in the time interval $\tau(I)$: as it is well known, it is fully defined with the distributions of all the random vectors $(Y(t_i))_{i\in K}$, with $(t_i)_{i\in K}\in\tau(I)^{|K|}$ and $K\subset I$, $0 < |K| < +\infty$.

$H_\Phi[\mathcal{L}(\underline{X}(I))]$ ($G_\Phi[\mathcal{L}(\underline{X}(I))]$) denotes the (un-)reliability of (C,Φ) during the time interval $\tau(I)$: $H_\Phi[\mathcal{L}(\underline{X}(I))] = P[\Phi(\underline{X}(t))=1; \forall t\in\tau(I)]$ ($G_\Phi[\mathcal{L}(\underline{X}(I))] = P[\Phi(\underline{X}(t))=0; \forall t\in\tau(I)]$

As soon as no confusion can arise, $H_\Phi[\mathcal{L}(\underline{X}(I))]$ and $G_\Phi[\mathcal{L}(\underline{X}(I))]$ are abbreviated as $H_\Phi(I)$ and $G_\Phi(I)$, respectively.

Remark 2.3: As in [20], two fundamental relations play a central part throughout the following:

(2.2) $H_{\Phi}[\mathcal{L}(\underline{X}(I))] + G_{\Phi}[\mathcal{L}(\underline{X}(I))] \leq 1;$

(2.3) $G_{\Phi}[\mathcal{L}(\underline{X}(I))] = H_{\Phi}D[\mathcal{L}(\underline{1}-\underline{X}(I))].$

The next theorem [20; Theorem 2.1] results immediately from (2.1).

Theorem 2.4: ("Bounds of Type 1-a" or "Min-Max Bounds")

For any binary broad-sense coherent system, (C,Φ), and for any closed interval I in $\mathbf{R}^+$,

(i) $L_{\Phi 1}(I) \leq H_{\Phi}(I) \leq U_{\Phi 1}(I)$ with $\begin{cases} L_{\Phi 1}(I) = \mathrm{Max}\{H_{\eta_r}(I) \;/\; r=1,\ldots,h\} \\ U_{\Phi 1}(I) = \min\{H_{\theta_r}(I) \;/\; r=1,\ldots,k\} \end{cases}$

(ii) $L'_{\Phi 1}(I) \leq G_{\Phi}(I) \leq U'_{\Phi 1}(I)$ with $\begin{cases} L'_{\Phi 1}(I) = \mathrm{Max}\{G_{\theta_r}(I) \;/\; r=1,\ldots,k\} \\ U'_{\Phi 1}(I) = \min\{G_{\eta_r}(I) \;/\; r=1,\ldots,h\} \end{cases}$

Remark 2.5: The "bounds" stated in the previous theorem can be said to be general since they do not require any special kind of dependence. Indeed, among the "basic bounds" considered in [20], three types of "bounds" can be distinguished with both of the following criteria: either, the dependence conditions under which they have been shown to be valid or the general procedure with which they have been put in evidence.

It is according to the second criterion above that they have been numbered here. Indeed, the notations [18] chosen for this study are quite different from the ones proposed in [20] but they appear convenient for dealing with the binary case as also with the multinary case in a concise and unambiguous way (see [16] and [18][19]).

Conventions 2.6: A "test-function" is any function such that any integral brought into play in the related context exists.

Definition 2.7: An R^m-valued random vector $(m \in N^*)$, $\underline{X}$, is (positively) **associated** if and only if for any couple of test-functions non-decreasing in each of their arguments from $\mathbf{R}^m$ into $\mathbf{R}$, (f,g), $\mathrm{Cov}(f(\underline{X}),g(\underline{X})) \geq 0$.

Remark 2.8: This well-known positive dependence has been introduced in [14] with various criteria and properties. The basic results which are necessary for the generalizations to be performed are briefly reviewed hereafter (see also [2]). Indeed, the previous definition is not altered when restricted to the class of the "test-functions" non-decreasing in each of their arguments from $\mathbf{R}^m$ into $\{0,1\}$, and consequently, association is stronger than the well-known positive (upper and lower) orthant dependences (or "weak association" [22]). This is the first reason why association has been playing a central part in **reliability** theory. The second reason is that it can be conserved through various transformations. In particular, the four basic properties listed in [20] remain sufficient for the generalizations to be performed: given two random vectors, $\underline{X}$ and $\underline{Y}$, respectively $\mathbf{R}^m$- and $\mathbf{R}^{m'}$-valued $((m,m')\in\mathbf{N}^{*2})$,

(A1) if $\underline{X} = (X_i)_{i=1,...,m}$ is associated, then, for every non-empty subset K of $\{j\in\mathbf{N}^* \ / \ j\leq m\}$, $(X_i)_{i\in K}$ is itself associated;

(A2) any real-valued random variable is associated;

(A3) if the random vectors $\underline{X}$ and $\underline{Y}$ are mutually independent and if each of them is associated, then, the $\mathbf{R}^{m+m'}$-valued random vector $(\underline{X},\underline{Y})$ is itself associated;

(A4) if $\underline{X}$ is associated, then, for any function f_i, $i = 1,...,m$, non-decreasing in each of its arguments from $\mathbf{R}^m$ into $\mathbf{R}$, the random vector $(f_i(\underline{X}))_{i=1,..,m}$ is itself associated.

Association can be easily extended in terms of stochastic processes with some "classical" procedure [13]:

Definition 2.9: For any closed interval I in $\mathbf{R}^+$,

(i) the stochastic processes $\{Y_i(t) \ / \ t\in\tau\}$, $i = 1,...,m$, are **mutually independent** if and only if for every finite non-empty subset K of $\mathbf{N}^*$ and for every $\underline{t} = (t_j)_{j\in K}\in\tau(I)^{|K|}$, the random vectors $(Y_i(t_j))_{j\in K}$, $i=1,...,m$, are mutually independent;

(ii) a stochastic process $\{Y(t) \ / \ t\in\tau\}$ is said to be **associated in the time interval $\tau(I)$** if and only if for every finite non-empty subset K of $\mathbf{N}^*$ and for every $\underline{t} = (t_i)_{i\in K}\in\tau(I)^{|K|}$, the random vector $(Y(t_i))_{i\in K}$ is associated.

Some criteria for time association of continuous time-parameter Markov chains have been proposed in [16]. The fundamental properties of association can be easily extended in terms of stochastic processes. Moreover, an essential result implicitely proved in [13] can be stated in general terms as follows.

Proposition 2.10: For any closed interval I in $\mathbf{R}^+$, if an S_C-valued stochastic process $\{\underline{X}(t) \overset{as}{=} (X_i(t))_{i \in C} \;/\; t \in \tau\}$ is associated in the time interval $\tau(I)$, then,

(i) $P[\bigcap_{i \in C} [X_i(t)=1;\ \forall t \in \tau(I)]] \geq \prod_{i \in C} P[X_i(t)=1;\ \forall t \in \tau(I)]$;

(ii) $P[\bigcap_{i \in C} [X_i(t)=0;\ \forall t \in \tau(I)]] \geq \prod_{i \in C} P[X_i(t)=0;\ \forall t \in \tau(I)]$.

The **proof** is omitted since this result can be obtained with the same procedure as the one proposed in [13] and equally used in various contexts in [20]. Indeed, this result follows immediately from the fact that association is stronger than both of the positive orthant dependences [2][22].

By (2.1), the following result [20; theorem 2.3] now can be obtained as a straight consequence of proposition 2.10.

Theorem 2.11: ("Bounds of Type 2-a" or "Minimal Path-Cut Sets Bounds") For any binary broad-sense coherent system, (C,Φ), and for any closed interval I in $\mathbf{R}^+$, if the joint performance process of its components is associated in the time interval $\tau(I)$, then,

(i) $L_{\Phi 2}(I) \leq H_\Phi(I) \leq U_{\Phi 2}(I)$

(ii) $L'_{\Phi 2}(I) \leq G_\Phi(I) \leq U'_{\Phi 2}(I)$

with: - $L_{\Phi 2}(I) = \prod_{r=1}^{k} H_{\theta_r}(I)$ and $L'_{\Phi 2}(I) = \prod_{r=1}^{h} G_{\gamma_r}(I)$

- $U_{\Phi 2}(I) = 1 - L'_{\Phi 2}(I)$ and $U'_{\Phi 2}(I) = 1 - L_{\Phi 2}(I)$

The next result has been proposed in [20; corollary 2.2].

Corollary 2.12: ("Bounds of Type 1-b" or "Min-Max Bounds Under Association") With the same assumptions as in theorem 2.11,

(i) $l_{\Phi 1}(\underline{p}(I)) \leq H_\Phi(I) \leq u_{\Phi 1}(\underline{q}(I))$

(ii) $l'_{\Phi 1}(\underline{q}(I)) \leq G_\Phi(I) \leq u'_{\Phi 1}(\underline{p}(I))$

with: - $l_{\Phi 1}(\underline{p}(I)) = \mathrm{Max}\{ \prod_{i \in H_r} p_i(I) \;/\; r=1,\ldots,h\}$;

- $l'_{\Phi 1}(\underline{q}(I)) = \mathrm{Max}\{ \prod_{i \in K_r} q_i(I) \;/\; r=1,\ldots,k\}$;

- $u_{\Phi 1}(\underline{q}(I)) = 1 - l'_{\Phi 1}(\underline{q}(I))$ and $u'_{\Phi 1}(\underline{p}(I)) = 1 - l_{\Phi 1}(\underline{p}(I))$.

Proof: As the previous theorem 2.11, this result can be deduced from proposition 2.10 (as an indication, see [20]). However, it can be shown in a shorter way as some straight consequence of both of the previous theorems [18]: in fact, according to (i) in theorem 2.4, it suffices to apply the "lower bounds" in (i) in theorem 2.11 to each series-system (H_r,η_r) characterized by its $|H_r|$ minimal cut sets, r=1,...,h, respectively: $H_{\eta_r}(I) \geq \prod_{i \in H_r} p_i(I)$. This yields the "lower bound" in (i). Then, the "lower bound" in (ii) follows immediately by duality (see (2.3)) while both of the "upper bounds" can be immediately obtained from the corresponding "lower bounds", by (2.2).

The next corollary [20; Corollary 2.4] follows immediately from theorem 2.11 and corollary 2.12.

Corollary 2.13: ("Bounds of Type 3-a" or "Mixed Bounds Under Association") With the same assumptions as in theorem 2.11,

(i) $L_{\Phi 3}(I) \leq H_{\Phi}(I) \leq U_{\Phi 3}(I)$

(ii) $L'_{\Phi 3}(I) \leq G_{\Phi}(I) \leq U'_{\Phi 3}(I)$

with: - $L_{\Phi 3}(I) = \mathrm{Max}\{l_{\Phi 1}(\underline{p}(I));L_{\Phi 2}(I)\}$;

- $L'_{\Phi 3}(I) = \mathrm{Max}\{l'_{\Phi 1}(\underline{q}(I));L'_{\Phi 2}(I)\}$;

- $U_{\Phi 3}(I) = 1 - L'_{\Phi 3}(I)$ and $U'_{\Phi 3}(I) = 1 - L_{\Phi 3}(I)$.

Both of the corollaries proposed hereafter constitute some slight generalizations of the results proposed in [20; Corollary 2.5]. Indeed, such extensions are necessary for the generalizations to be performed in terms of pseudo-modular decompositions.

Notations 2.14: For every $\underline{u} = (u_i)_{i=1,...,m} \in [0,1]^m$ ($m \in \mathbf{N}^*$) and for every non-empty subset I of $\{j \in \mathbf{N}^* \ / \ j \leq m\}$, $\coprod_{i \in I} u_i = 1 - \prod_{i \in I} (1-u_i)$.

Corollary 2.15: ("Bounds of Type 2-b" or "Minimal Path-Cut Sets Bounds" With Some Further Conditions of Independence) For any binary broad-sense coherent system, (C,Φ), and for any closed interval I in $\mathbf{R}^+$, if the joint performance process of all the components of (C,Φ), $\{\underline{X}(t) \ / \ t \in \tau\}$, is associated in the time interval $\tau(I)$ and if for each minimal path (cut) set of (C,Φ), $H \in \mathcal{H}(\Phi)$ ($K \in \mathcal{K}(\Phi)$), the marginal performance processes $\{X_i(t) \ / \ t \in \tau\}$, $i \in H$ ($i \in K$), are mutually independent in the same interval of time, then,

(i) $l_{\Phi 2}(\underline{p}(I)) \leq H_{\Phi}(I) \leq u_{\Phi 2}(\underline{q}(I))$

(ii) $l'_{\Phi 2}(\underline{q}(I)) \leq G_{\Phi}(I) \leq u'_{\Phi 2}(\underline{p}(I))$

with: - $l_{\Phi 2}(\underline{p}(I)) = \prod_{r=1}^{k} \coprod_{i \in K_r} p_i(I)$ and $l'_{\Phi 2}(\underline{q}(I)) = \prod_{r=1}^{h} \coprod_{i \in H_r} q_i(I)$;

- $u_{\Phi 2}(\underline{q}(I)) = 1 - l'_{\Phi 2}(\underline{q}(I))$ and $u'_{\Phi 2}(\underline{p}(I)) = 1 - l_{\Phi 2}(\underline{p}(I))$.

The **proof** is omitted: it suffices to note that the dependence conditions stated above are sufficient for the proof proposed in [20; Corollary 2.5].

The next result follows immediately.

Corollary 2.16: ("Bounds of Type 3-b" or "Mixed Bounds" With Some Further Conditions of Independence) With the same assumptions as in corollary 2.15,

(i) $l_{\Phi 3}(\underline{p}(I)) \leq H_{\Phi}(I) \leq u_{\Phi 3}(\underline{q}(I))$

(ii) $l'_{\Phi 3}(\underline{q}(I)) \leq G_{\Phi}(I) \leq u'_{\Phi 3}(\underline{p}(I))$

with: - $l_{\Phi 3}(\underline{p}(I)) = \text{Max}\{l_{\Phi 1}(\underline{p}(I)); l_{\Phi 2}(\underline{p}(I))\}$;

- $l'_{\Phi 3}(\underline{q}(I)) = \text{Max}\{l'_{\Phi 1}(\underline{q}(I)); l'_{\Phi 2}(\underline{q}(I))\}$;

- $u_{\Phi 3}(\underline{q}(I)) = 1 - l'_{\Phi 3}(\underline{q}(I))$ and $u'_{\Phi 3}(\underline{p}(I)) = 1 - l_{\Phi 3}(\underline{p}(I))$.

Notations 2.17: As soon as some confusion can arise, the "bounds" $L_{\Phi e}(I)$, $L'_{\Phi e}(I)$, $U_{\Phi e}(I)$ and $U'_{\Phi e}(I)$, for every $e = 1,2,3$, will be designated, in a more detailed way, as $L_{\Phi e}[\mathcal{L}(\underline{X}(I))]$, $L'_{\Phi e}[\mathcal{L}(\underline{X}(I))]$, $U_{\Phi e}[\mathcal{L}(\underline{X}(I))]$ and $U'_{\Phi e}[\mathcal{L}(\underline{X}(I))]$, respectively. This must be done at once in the next remark.

Remark 2.18: As in [20], some relations of duality which result immediately from (2.3) will appear helpful in what follows:

(2.4) $U'_{\Phi 1}[\mathcal{L}(\underline{X}(I))] = U_{\Phi^D 1}[\mathcal{L}(\underline{1}-\underline{X}(I))]$;

(2.5) for every $e = 1,2,3$, $L'_{\Phi e}[\mathcal{L}(\underline{X}(I))] = L_{\Phi^D e}[\mathcal{L}(\underline{1}-\underline{X}(I))]$;

(2.6) for every $e= 1,2,3$, $l'_{\Phi e}(\underline{q}(I)) = l_{\Phi^D e}(\underline{q}(I))$.

Furthermore, as it has been noted in [20], the "upper bounds" stated from theorem 2.11 to corollary 2.16 may turn out to be "poor" since they are only deduced from the related "lower bounds" by (2.2).

3. PSEUDO-MODULAR DECOMPOSITIONS AND "REFINED BOUNDS" FOR THE INTERVAL RELIABILITY AND UNRELIABILITY:

The general principle of the "bounds" refinements proposed in terms of modular decompositions in [5] and equally applied in [20] can be extended, in a straight way, in terms of pseudo-modular decompositions: the "refined bounds" proposed in this study are obtained by applying the "bounds" of type e-b, e = 1, 2 or 3, to the organizing system of the concerned pseudo-modular decomposition and the "bounds" of the same type or of some other type to the binary broad-sense coherent systems which compose its pseudo-modules.
Indeed, it can be easily checked that all the "basic bounds" can be extended in terms of any collection of path (cut) sets of the concerned system which contains all its minimal path (cut) sets; in addition, the general principle stated above can be applied with any decomposition which is defined with some binary coherent systems and which allows to express the minimal path (cut) sets of the complete system as some unions of the minimal path (cut) sets of its subsystems. So, by theorem A.3.10, it could be applied with any simple coherent decomposition. However, the best "basic bounds" are obtained with the collection of the minimal path (cut) sets of the complete system, only, and by remark A.3.21 and corollary A.3.18, they must be refined by considering two pseudo-modular decompositions of (C,Φ) simultaneously: one with respect to $\mathcal{H}(\Phi)$, the other one with respect to $\mathcal{K}(\Phi)$.

Assumptions 3.1: For every $c\in\mathbf{N}^*$ and every $(c_r)_{r=1,..,c}\in\mathbf{N}^{*c}$, let $U(c;(c_r)_r) = \bigcup_{r=1}^{c} \{r\}\times\{j\in\mathbf{N}^* \,/\, j\leq c_r\}$.
Let $(a,b)\in\mathbf{N}^{*2}$, $(a_r)_{r=1,..,a}\in\mathbf{N}^{*a}$ and $(b_r)_{r=1,..,b}\in\mathbf{N}^{*b}$.
Throughout the following, $\mathcal{A} = \{\{(A_r,\alpha_{rv_r}) \,/\, v_r=1,..,a_r\} \,/\, r=1,..,a\}$ ($\mathcal{B} = \{\{(B_r,\beta_{rv_r}) \,/\, v_r=1,..,b_r\} \,/\, r=1,..,b\}$) denotes a pseudo-modular decomposition of some binary broad-sense coherent system, (C,Φ), with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) and with an organizing system $(A=U(a;(a_r)_r),\alpha)$ ($(B=U(b;(b_r)_r),\beta)$):

$$(3.1)\quad \forall t\in\tau,\ \Phi(\underline{X}(t)) \overset{as}{=} \alpha((\alpha_{rv_r}(\underline{X}_{A_r}(t)))_{(r,v_r)\in A}) \overset{as}{=} \beta((\beta_{rv_r}(\underline{X}_{B_r}(t)))_{(r,v_r)\in B}).$$

Consequently, for any closed interval I in $\mathbf{R}^+$,

$$(3.2)\quad H_\Phi[\mathcal{L}(\underline{X}(I))] = H_\alpha[\mathcal{L}((\alpha_{rv_r}(\underline{X}_{A_r}(I)))_{(r,v_r)\in A})] = H_\beta[\mathcal{L}((\beta_{rv_r}(\underline{X}_{B_r}(I)))_{(r,v_r)\in B})]$$

And obviously, an analogous relation holds in terms of the interval unreliabilities.

Notations 3.2: According to the general approach to be applied, some vectors of "bounds" will appear convenient throughout the following: for every $e\in\{1,2,3\}$ and for every closed interval I in $\mathbf{R}^+$,

$\underline{D}_{\alpha e}(I) = (D_{\alpha_{rv_r}}{}^e(I))_{(r,v_r)\in A}$ and $\underline{D}_{\beta e}(I) = (D_{\beta_{rv_r}}{}^e(I))_{(r,v_r)\in B}$ where D is only a generic term for L, L', U and U' (see Section 2).

$\underline{d}_{\alpha e} \equiv (d_{\alpha_{rv_r}}{}^e)_{(r,v_r)\in A}$ and $\underline{d}_{\beta e} \equiv (d_{\beta_{rv_r}}{}^e)_{(r,v_r)\in B}$ where d is only a generic term for l, l', u and u' (see Section 2).

In an analogous way,

$\underline{H}_\alpha(I) = (H_{\alpha_{rv_r}}(I))_{(r,v_r)\in A}$ and $\underline{G}_\alpha(I) = (G_{\alpha_{rv_r}}(I))_{(r,v_r)\in A}$,

$\underline{H}_\beta(I) = (H_{\beta_{rv_r}}(I))_{(r,v_r)\in B}$ and $\underline{G}_\beta(I) = (G_{\beta_{rv_r}}(I))_{(r,v_r)\in B}$.

Remark 3.3: It appears always helpful to establish some easy "bridge" between the results to be generalized and the generalizations to be performed; such a "bridge" can be easily put in evidence between the modular decompositions and the simple coherent subsystems (and consequently, the pseudo-modular decompositions). Let $\mathcal{K}(\beta) = \{K'_r \ / \ r=1,\dots,k'\}$ and for every $r = 1,\dots,k'$ respectively, let (C,Φ_r) be the binary coherent system defined as follows: for every $\underline{x}\in S_C$,

$$\Phi_r(\underline{x}) = \xi_r((\beta_{sv_s}(\underline{x}_{B_s}))_{sv_s}) = \coprod_{(s,v_s)\in K'_r} \beta_{sv_s}(\underline{x}_{B_s});$$

by definition A.3.5, $\{(B_s,\beta_{sv_s}) \ / \ (s,v_s)\in K'_r\}$ is a modular decomposition of (C,Φ_r) with a parallel organizing system, (K'_r,ξ_r). In addition, by (3.1) and corollary A.3.18,

$$(3.3)\qquad \Phi \equiv \prod_{r=1}^{k'} \Phi_r \quad\text{and}\quad \mathcal{K}(\Phi) = \bigcup_{r=1}^{k'} \mathcal{K}(\Phi_r).$$

Of course, some analogous result can be equally obtained from the minimal path sets of (A,α).

The next result has been proposed in [20; Lemma A.1.1].

Lemma 3.4: Let $\{(M_r,\mu_r) \ / \ r=1,\ldots,m\}$ ($m\in\mathbf{N}^*$) be a modular decomposition for some binary coherent system, (C,Φ), with a parallel organizing system; for any closed interval I in $\mathbf{R}^+$, if the performance processes of the modules, $\{\mu_r(\underline{X}_{M_r}(t)) \ / \ t\in\tau\}$, $r = 1,\ldots,m$, are mutually independent in the time interval $\tau(I)$, then,

$$U_{\Phi 1}(I) \leq 1 - \prod_{r=1}^{m} L'_{\mu_r 1}(I) \quad \text{and} \quad L'_{\Phi 1}(I) = \prod_{r=1}^{m} L'_{\mu_r 1}(I).$$

The next theorem and its corollaries extend "theorem 2.6" in [20].

Theorem 3.5: ("Composition of the Bounds of Type 1")

Under assumptions 3.1, for any closed interval I in $\mathbf{R}^+$, if

(a) for each minimal path set of (A,α), $H\in\mathcal{H}(\alpha)$, the performance processes $\{\alpha_{rv_r}(\underline{X}_{A_r}(t)) \ / \ t\in\tau\}$, $(r,v_r)\in H$, are mutually independent in the time interval $\tau(I)$,

(b) for each minimal cut set of (B,β), $K\in\mathcal{K}(\beta)$, the performance processes $\{\beta_{rv_r}(\underline{X}_{B_r}(t)) \ / \ t\in\tau\}$, $(r,v_r)\in K$, are mutually independent in the time interval $\tau(I)$,

(c) the joint performance processes of all the pseudo-modules in $\mathcal{A}$, $\{(\alpha_{rv_r}(\underline{X}_{A_r}(t)))_{(r,v_r)\in A} \ / \ t\in\tau\}$, is associated in the same interval of time,

(d) the joint performance processes of all the pseudo-modules in $\mathcal{B}$, $\{(\beta_{rv_r}(\underline{X}_{B_r}(t)))_{(r,v_r)\in B} \ / \ t\in\tau\}$, is associated in the same interval of time,

$$L_{\Phi 1}(I) \overset{1}{=} l_{\alpha 1}(\underline{\underline{L}}_{\alpha 1}(I)) \overset{2}{\leq} l_{\alpha 1}(\underline{H}_{\alpha}(I)) \overset{3}{\leq} H_{\Phi}(I) \overset{4}{\leq} u_{\beta 1}(\underline{G}_{\beta}(I)) \overset{5}{\leq} u_{\beta 1}(\underline{\underline{L}}'_{\beta 1}(I))$$

$$U_{\Phi 1}(I) \overset{6}{\leq} 1 - l'_{\beta 1}(\underline{\underline{L}}'_{\beta 1}(I))$$

$$L'_{\Phi 1}(I) \overset{7}{=} l'_{\beta 1}(\underline{\underline{L}}'_{\beta 1}(I)) \overset{8}{\leq} l'_{\beta 1}(\underline{G}_{\beta}(I)) \overset{9}{\leq} G_{\Phi}(I) \overset{10}{\leq} u'_{\alpha 1}(\underline{H}_{\alpha}(I)) \overset{11}{\leq} u'_{\alpha 1}(\underline{\underline{L}}_{\alpha 1}(I))$$

$$U'_{\Phi 1}(I) \overset{12}{\leq} 1 - l_{\alpha 1}(\underline{\underline{L}}_{\alpha 1}(I)).$$

Proof: As "theorem 2.6" in [20], this result can be proved by means of lemma 3.4. First, note for each $r = 1,\ldots,k'$, the modular decomposition of the binary coherent system (C,Φ_r) defined in Remark 3.3 satisfies the assumptions of lemma 3.4; consequently,

$U_{\Phi 1}(I) = \min\{H_{\theta_r}(I) \ / \ r=1,\ldots,k\}$ (by definition; see Theorem 2.4)

$= \min\{U_{\Phi_r 1}(I) \;/\; r=1,\ldots,k'\}$ (by (3.3))

$\leq \min\{1 - \prod_{(s,v_s)\in K'_r} L'_{\beta_{sv_s}1}(I) \;/\; r=1,\ldots,k'\}$ (by Lemma 3.4)

$\leq 1 - l'_{\beta 1}(\underline{L}'_{\beta 1}(I))$ (by definition of $l'_{\beta 1}$; see Corollary 2.12).

The inequality 7 can be shown by following an analogous approach:

$L'_{\Phi 1}(I) = \mathrm{Max}\{G_{\Theta_r}(I) \;/\; r=1,\ldots,k\}$ (by definition; see Theorem 2.4)

$= \mathrm{Max}\{L'_{\Phi_r 1}(I) \;/\; r=1,\ldots,k'\}$ (by (3.3))

$= \mathrm{Max}\{\prod_{(s,v_s)\in K'_r} L'_{\beta_{sv_s}1}(I) \;/\; r=1,\ldots,k'\}$ (by Lemma 3.4)

$= l'_{\beta 1}(\underline{L}'_{\beta 1}(I))$ (by definition of $l'_{\beta 1}$).

Furthermore, the non-decreasing (non-increasing) property of $l'_{\beta 1}$ ($u'_{\alpha 1}$) in each of its arguments yields the inequality 8 (11) while by (3.1), the inequalities 9 and 10 can be immediately obtained by applying the "bounds of type 1-b" to the organizing systems (B,β) and (A,α), respectively (see Corollary 2.12).

The inequality 12 and the inequalities 1 to 5 now follow immediately by duality (see proposition A.3.16, (2.3) and remark 2.18). As an indication, the equality 1 can be deduced from the equality 7 as follows (see Proposition A.3.16):

$L_{\Phi 1}[\mathcal{L}(\underline{X}(I))] = L'_{\Phi}D_1[\mathcal{L}(\underline{1}-\underline{X}(I))]$ (by (2.5))

$= l'_{\alpha}D_1(\underline{L}'_{\alpha}D_1[\mathcal{L}(\underline{1}-\underline{X}(I))])$ (by the inequality 7)

$= l_{\alpha 1}(\underline{L}_{\alpha 1}[\mathcal{L}(\underline{X}(I))])$ (by (2.5) and (2.6)) $\square$

The following particular case is immediate (see Definition A.3.4).

Corollary 3.6: Under assumptions 3.1, for any closed interval I in $\mathbf{R}^+$, all the inequalities in theorem 3.5 hold if the following conditions of dependence are satisfied:

(a) the performance processes of the pseudo-modules with respect to $\mathcal{H}(\Phi)$, $\{(\alpha_{rv_r}(\underline{X}_{A_r}(t)))_{v_r} \;/\; t\in\tau\}$, $r = 1,\ldots,a$, are mutually independent in the time interval $\tau(I)$,

(b) the performance processes of the pseudo-modules with respect to $\mathcal{K}(\Phi)$, $\{(\beta_{rv_r}(\underline{X}_{B_r}(t)))_{v_r} \;/\; t\in\tau\}$, $r = 1,\ldots,b$, are mutually independent in the time interval $\tau(I)$,

(c) each of these performance processes is associated in the same interval of time.

Remark 3.7: Some special care is needed for the "refined bounds" defined with the "basic bounds of type 3-b": each of these "bounds"

relies upon the knowledge of $\mathcal{H}(\Phi)$ and of $\mathcal{K}(\Phi)$ and consequently should be refined both in terms of $\mathcal{A}$ and $\mathcal{B}$ with some extended definitions: $l_{\alpha\beta 3} \equiv \mathrm{Max}\{l_{\alpha 1}, l_{\beta 2}\}$; $l'_{\alpha\beta 3} \equiv \mathrm{Max}\{l'_{\alpha 2}, l'_{\beta 1}\}$;
$u_{\alpha\beta 3} \equiv \min\{u_{\alpha 2}, u_{\beta 1}\} \equiv 1 - l'_{\alpha\beta 3}$;
and in an analogous way, $u'_{\alpha\beta 3} \equiv 1 - l_{\alpha\beta 3}$.

Corollary 3.8: ("Composition of the Bounds of Type 1-a and 3-b")
With the same assumptions as in theorem 3.5 or as in corollary 3.6,

$$L_{\Phi 1}(I) \overset{1}{\le} l_{\alpha\beta 3}(\underline{L}_{\alpha 1}(I), \underline{L}_{\beta 1}(I)) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha}(I), \underline{H}_{\beta}(I)) \overset{3}{\le} H_{\Phi}(I)$$

$$H_{\Phi}(I) \overset{4}{\le} u_{\alpha\beta 3}(\underline{G}_{\alpha}(I), \underline{G}_{\beta}(I)) \overset{5}{\le} u_{\alpha\beta 3}(\underline{L}'_{\alpha 1}(I), \underline{L}'_{\beta 1}(I))$$

$$L'_{\Phi 1}(I) \overset{6}{\le} l'_{\alpha\beta 3}(\underline{L}'_{\alpha 1}(I), \underline{L}'_{\beta 1}(I)) \overset{7}{\le} l'_{\alpha\beta 3}(\underline{G}_{\alpha}(I), \underline{G}_{\beta}(I)) \overset{8}{\le} G_{\Phi}(I)$$

$$G_{\Phi}(I) \overset{9}{\le} u'_{\alpha\beta 3}(\underline{H}_{\alpha}(I), \underline{H}_{\beta}(I)) \overset{10}{\le} u'_{\alpha\beta 3}(\underline{L}_{\alpha 1}(I), \underline{L}_{\beta 1}(I)).$$

Proof: Note the assumptions are sufficient for applying the "bounds" thus brought into play. According to the definition of $l_{\alpha\beta 3}$, the inequality 1 can be immediately deduced from the equality 1 in theorem 3.5: $L_{\Phi 1}(I) = l_{\alpha 1}(\underline{L}_{\alpha 1}(I)) \le \mathrm{Max}\{l_{\alpha 1}(\underline{L}_{\alpha 1}(I)), l_{\beta 2}(\underline{L}_{\beta 1}(I))\}$.
The non-decreasing property of $l_{\alpha\beta 3}$ in each of its arguments yields the inequality 2. By (3.2), the inequality 3 can be immediately obtained by applying the "bounds" of type 1-b to (A,α) (see Corollary 2.12) and the "bounds" of type 2-b to (B,β) (see Corollary 2.15): $l_{\alpha 1}(\underline{H}_{\alpha}(I)) \le H_{\alpha}(I)$ and $l_{\beta 2}(\underline{H}_{\beta}(I)) \le H_{\beta}(I)$. The inequalities 6 to 8 now follow immediately by duality (see Proposition A.3.16, (2.3) and remark 2.18) and by (2.2) yield the inequalities 4 and 5, which in their turns yield the inequalities 9 and 10, by duality.

If some stronger conditions of dependence hold, the "composition of the bounds of type 3" can be partly applied and yields some improved "bounds" : this is shown with the next result.

Corollary 3.9: Under assumptions 3.1, for any closed interval I in $\mathbf{R}^+$, if all the conditions of dependence stated in theorem 3.5 (or, in particular, the conditions (a) and (b) in corollary 3.6) are satisfied in the time interval $\tau(I)$ and if in addition, the joint performance process of the components of each pseudo-module, $\{\underline{X}_{A_r}(t) \;/\; t\in\tau\}$, $r = 1,\ldots,a$, $\{\underline{X}_{B_r}(t) \;/\; t\in\tau\}$, $r = 1,\ldots,b$, is associated in the same interval of time, then, for every $\underline{v}\in\{\underline{L}'_{\alpha e}(I)\ /\ e=1,2,3\}\cup\{\underline{l}'_{\alpha 1}(\underline{g}(I))\}$ and for every $\underline{w}\in\{\underline{L}_{\beta e}(I)\ /\ e=1,2,3\}\cup\{\underline{l}_{\beta 1}(\underline{p}(I))\}$,

$$L_{\Phi 1}(I) \overset{1}{\le} l_{\alpha\beta 3}(\underline{L}_{\alpha 1}(I),\underline{w}) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha}(I),\underline{H}_{\beta}(I)) \overset{3}{\le} H_{\Phi}(I)$$

$$H_{\Phi}(I) \overset{4}{\le} u_{\alpha\beta 3}(\underline{G}_{\alpha}(I),\underline{G}_{\beta}(I)) \overset{5}{\le} u_{\alpha\beta 3}(\underline{v},\underline{L}'_{\beta 1}(I))$$

$$L'_{\Phi 1}(I) \overset{6}{\le} l'_{\alpha\beta 3}(\underline{v},\underline{L}'_{\beta 1}(I)) \overset{7}{\le} l'_{\alpha\beta 3}(\underline{G}_{\alpha}(I),\underline{G}_{\beta}(I)) \overset{8}{\le} G_{\Phi}(I)$$

$$G_{\Phi}(I) \overset{9}{\le} u'_{\alpha\beta 3}(\underline{H}_{\alpha}(I),\underline{H}_{\beta}(I)) \overset{10}{\le} u'_{\alpha\beta 3}(\underline{L}_{\alpha 1}(I),\underline{w}).$$

Proof: First, note the assumptions are sufficient for applying all the "bounds" thus brought into play. The inequality 1 follows immediately from the equality 1 in theorem 3.5: as an indication,

$L_{\Phi 1}(I) = l_{\alpha 1}(\underline{L}_{\alpha 1}(I)) \le \mathrm{Max}\{l_{\alpha 1}(\underline{L}_{\alpha 1}(I)), l_{\beta 2}(\underline{w})\}$.

The inequality 2 follows immediately from the non-decreasing property of $l_{\alpha\beta 3}$: according to the range assigned to $\underline{w}$, for every $(r,v_r)\in B$, $w_{rv_r} \le H_{\beta_{rv_r}}(I)$. The inequality 3 has been proved with corollary 3.8. All the other inequalities can be easily deduced from the three first ones with some arguments quite analogous to the ones proposed for the inequalities with the same numbers in corollary 3.8.

Indeed, some stronger result holds under the assumptions of corollary 3.9; this is shown with the next theorem and its corollary; these results extend "theorem 2.7" proposed in [20] and can be proved with the following lemma proposed in [20; Lemma A.2.1].

Lemma 3.10: If $\{(M_r,\mu_r)\ /\ r=1,\dots,m\}$ $(m\in\mathbf{N}^*)$ is a modular decomposition for some binary coherent system, (C,Φ), with a parallel organizing system, then, $l'_{\Phi 1}(\underline{g}(I)) = \prod_{r=1}^{m} l'_{\mu_r 1}(\underline{g}(I))$.

Theorem 3.11: ("Composition of the Bounds of Type 1-b")

With the same assumptions as in corollary 3.9,

$$l_{\Phi 1}(\underline{p}(I)) \overset{1}{=} l_{\alpha 1}(\underline{l}_{\alpha 1}(\underline{p}(I))) \overset{2}{\le} l_{\alpha 1}(\underline{H}_{\alpha}(I)) \overset{3}{\le} H_{\Phi}(I)$$

$$H_{\Phi}(I) \overset{4}{\le} u_{\beta 1}(\underline{G}_{\beta}(I)) \overset{5}{\le} u_{\beta 1}(\underline{l}'_{\beta 1}(\underline{g}(I))) \overset{6}{=} u_{\Phi 1}(\underline{g}(I))$$

$$l'_{\Phi 1}(\underline{g}(I)) \overset{7}{=} l'_{\beta 1}(\underline{l}'_{\beta 1}(\underline{g}(I))) \overset{8}{\le} l'_{\beta 1}(\underline{G}_{\beta}(I)) \overset{9}{\le} G_{\Phi}(I)$$

$$G_{\Phi}(I) \overset{10}{\le} u'_{\alpha 1}(\underline{H}_{\alpha}(I)) \overset{11}{\le} u'_{\alpha 1}(\underline{l}_{\alpha 1}(\underline{p}(I))) \overset{12}{=} u'_{\Phi 1}(\underline{p}(I))$$

Proof: First, note the assumptions are sufficient for applying all the "bounds of type 1-b" brought into play. Indeed, the main part of the proof concerns the equality 7; it can be proved by applying lemma 3.10 to the modular decompositions of the binary coherent systems (C,Φ_r), $r=1,\dots,k'$, defined in remark 3.3:

$$\begin{aligned} l'_{\Phi 1}(\underline{q}(I)) &= \mathrm{Max}\{\prod_{i\in K_r} q_i(I) \;/\; r=1,\dots,k\} && \text{(by definition)}\\ &= \mathrm{Max}\{l'_{\Phi_r 1}(\underline{q}(I)) \;/\; r=1,\dots,k'\} && \text{(by (3.3))}\\ &= \mathrm{Max}\{\prod_{(s,v_s)\in K'_r} l'_{\beta_{s v_s} 1}(\underline{q}(I)) \;/\; r=1,\dots,k'\} && \text{(by Lemma 3.10)}\\ &= l'_{\beta 1}(\underline{l}'_{\beta 1}(\underline{q}(I))) && \text{(by definition of } l'_{\beta 1}). \end{aligned}$$

The non-decreasing property of $l'_{\beta 1}$ in each of its arguments yields the inequality 8 while the inequality 9 has been proved with theorem 3.5. The inequalities 1 to 3 follow immediately from the inequalities 7 to 9 by duality and by (2.2) yield the inequalities 10 to 12, which in their turn yield the inequalities 4 to 6, by duality.

Corollary 3.12: With the same assumptions as in corollary 3.9, for every $\underline{v}\in\{\underline{L}'_{\alpha e}(I) \;/\; e=1,2,3\}\cup\{\underline{l}'_{\alpha 1}(\underline{q}(I))\}$ and for every $\underline{w}\in\{\underline{L}_{\beta e}(I) \;/\; e=1,2,3\}\cup\{\underline{l}_{\beta 1}(\underline{p}(I))\}$,

$$l_{\Phi 1}(\underline{p}(I)) \overset{1}{\le} l_{\alpha\beta 3}(\underline{L}_{\alpha 3}(I),\underline{w}) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha}(I),\underline{H}_{\beta}(I)) \overset{3}{\le} H_{\Phi}(I)$$
$$H_{\Phi}(I) \overset{4}{\le} u_{\alpha\beta 3}(\underline{G}_{\alpha}(I),\underline{G}_{\beta}(I)) \overset{5}{\le} u_{\alpha\beta 3}(\underline{v},\underline{L}'_{\beta 3}(I)) \overset{6}{\le} u_{\Phi 1}(\underline{q}(I))$$
$$l'_{\Phi 1}(\underline{q}(I)) \overset{7}{\le} l'_{\alpha\beta 3}(\underline{v},\underline{L}'_{\beta 3}(I)) \overset{8}{\le} l'_{\alpha\beta 3}(\underline{G}_{\alpha}(I),\underline{G}_{\beta}(I)) \overset{9}{\le} G_{\Phi}(I)$$
$$G_{\Phi}(I) \overset{10}{\le} u'_{\alpha\beta 3}(\underline{H}_{\alpha}(I),\underline{H}_{\beta}(I)) \overset{11}{\le} u'_{\alpha\beta 3}(\underline{L}_{\alpha 3}(I),\underline{w}) \overset{12}{\le} u'_{\Phi 1}(\underline{p}(I))$$

Proof: The inequality 1 follows immediately from the equality 1 in theorem 3.11: by the non-decreasing property of $l_{\alpha 1}$,

$$l_{\Phi 1}(\underline{p}(I)) \le \mathrm{Max}\{l_{\alpha 1}(\underline{l}_{\alpha 1}(\underline{p}(I))),l_{\beta 2}(\underline{w})\} \le \mathrm{Max}\{l_{\alpha 1}(\underline{L}_{\alpha 3}),l_{\beta 2}(\underline{w})\}.$$

The non-decreasing property of $l_{\alpha\beta 3}$ (i.e. of $l_{\alpha 1}$ and of $l_{\beta 2}$) in each of its arguments yields immediately the inequality 2 while the inequality 3 has been proved with corollary 3.8. All the other inequalities can be easily checked from the three first ones with some arguments analogous to the ones proposed for the inequalities with the same numbers in theorem 3.11.

Remark 3.13: Of course, if in addition, for every $(r,v_r)\in A$ ($(r,v_r)\in B$) and for every $H\in\mathcal{H}(\alpha_{r v_r})$ ($K\in\mathcal{K}(\beta_{r v_r})$), the marginal performance processes $\{X_i(t) \;/\; t\in\tau\}$, $i\in H$ ($i\in K$), are mutually independent in the time interval $\tau(I)$, the inequalities in corollaries 3.9 and 3.12 equally hold for every $\underline{v}\in\{\underline{1}'_{\alpha e}(\underline{g}(I)) \ / \ e=2,3\}$ $\underline{w}\in\{\underline{1}_{\beta e}(\underline{p}(I)) \ / \ e=2,3\})$ (see Corollaries 2.15 and 2.16).

4. PSEUDO-MODULAR DECOMPOSITIONS AND "REFINED BOUNDS" FOR THE AVAILABILITY:

Throughout this section, "bounds" for the (instantaneous) availability (i.e. at some fixed point in time) are refined in terms of pseudo-modular decompositions. Indeed, if all the components of a binary broad-sense coherent system "have a life" (i.e. their respective performance processes are almost-surely non-increasing), the system itself "has a life" [10] and its availability function is identical to its reliability function. Consequently, it should be retained throughout the following that all the "bounds" for the availability concern also the particular case of the reliability for the binary coherent systems all the components of which "have a life". In addition, this particular case can be treated, in an alternative way, in terms of life lengths and life functions (see Remark A.2.15 and Definition A.2.16).

Remark 4.1: Throughout the following, a binary broad-sense coherent system, (C,Φ), is "observed" at some fixed point in time $t\in\tau$ only (i.e. $\tau(I) = I\cap\tau = [t,t]$); so, any time reference can be omitted and it suffices to retain the performance vector of its components at the concerned point in time, $\underline{X} = \underline{X}(t)$. Let H_Φ (G_Φ) be the corresponding availability (unavailability) of (C,Φ): $H_\Phi = P[\Phi(\underline{X})=1]$ ($G_\Phi = P[\Phi(\underline{X})=0]$). So, $H_\Phi+G_\Phi = 1$ and it suffices to consider the "bounds" for the availability, for instance.
With these conventions, the "basic bounds" for the availability can be expressed as follows [2]:

$$L_{\Phi 1} = \mathrm{Max}\{H_{\eta_r} \ / \ r=1,\ldots,h\}; \ U_{\Phi 1} = \min\{H_{\theta_r} \ / \ r=1,\ldots,k\};$$

$$L_{\Phi 2} = \prod_{r=1}^{k} H_{\theta_r}; \ U_{\Phi 2} = \coprod_{r=1}^{h} H_{\eta_r};$$

$$l_{\Phi 1}(\underline{p}) = \mathrm{Max}\{\prod_{i\in H_r} p_i \ / \ r=1,\ldots,h\}; \ u_{\Phi 1}(\underline{p}) = \min\{\coprod_{i\in K_r} p_i \ / \ r=1,\ldots,k\};$$

$$l_{\Phi 2}(\underline{p}) = \prod_{r=1}^{k} \coprod_{i\in K_r} p_i; \ u_{\Phi 2}(\underline{p}) = \coprod_{r=1}^{h} \prod_{i\in H_r} p_i;$$

where $\underline{p} = (p_i)_{i\in C}\in[0,1]^{|C|}$ is the vector of the components availabilities. The corresponding expressions of the "bounds" of type 3 are immediate. In particular, it should be noted that the upper "bounds" of type e-b, e = 1, 2 or 3, are defined with some different arguments (i.e. $\underline{p} = \underline{1} - g([t,t])$; see Notations 2.2).

Both of the following corollaries extend "theorem 3.1" proposed in [20]. In particular, the next result follows immediately from corollary 3.8: the assumptions of theorem 3.5 are satisfied in the time interval $\tau(I) = [t,t]$.

Corollary 4.2: ("Composition of the Bounds of Type 1 and 3")
Under assumptions 3.1, if

(a) for each minimal path set of (A,α), $H \in \mathcal{H}(\alpha)$, the performance variables $\alpha_{rv_r}(\underline{X}_{A_r})$, $(r,v_r) \in H$, are mutually independent,

(b) for each minimal cut set of (B,β), $K \in \mathcal{K}(\beta)$, the performance variables $\beta_{rv_r}(\underline{X}_{B_r})$, $(r,v_r) \in K$, are mutually independent,

(c) the joint performance vector of all the pseudo-modules in $\mathcal{A}$, $(\alpha_{rv_r}(\underline{X}_{A_r}))_{(r,v_r)\in A}$ is associated,

(d) the joint performance vector of all the pseudo-modules in $\mathcal{B}$., $(\beta_{rv_r}(\underline{X}_{B_r}))_{(r,v_r)\in B}$ is associated,

then, $L_{\Phi 1} \overset{1}{\le} l_{\alpha\beta 3}(\underline{L}_{\alpha 1},\underline{L}_{\beta 1}) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha},\underline{H}_{\beta}) \overset{3}{\le} H_{\Phi}$

$H_{\Phi} \overset{4}{\le} u_{\alpha\beta 3}(\underline{H}_{\alpha},\underline{H}_{\beta}) \overset{5}{\le} u_{\alpha\beta 3}(\underline{U}_{\alpha 1},\underline{U}_{\beta 1}) \overset{6}{\le} U_{\Phi 1}$.

Throughout the following, h_{α} and h_{β} denote the performance functions of the organizing systems (A,α) and (B,β), respectively.

Corollary 4.3: Under assumptions 3.1, all the inequalities stated in corollary 4.2 still hold if the following conditions of dependence are satisfied:

(4.1) the performance vectors of the pseudo-modules with respect to $\mathcal{H}(\Phi)$, $(\alpha_{rv_r}(\underline{X}_{A_r}))_{v_r}$, $r = 1,\dots,a$, are mutually independent;

(4.2) the performance vectors of the pseudo-modules with respect to $\mathcal{K}(\Phi)$, $(\beta_{rv_r}(\underline{X}_{B_r}))_{v_r}$, $r = 1,\dots,b$, are mutually independent;

(4.3) each of these performance vectors is associated.

If in addition, the pseudo-modular decompositions $\mathcal{A}$ and $\mathcal{B}$ are monotone, $l_{\alpha\beta 3}(\underline{L}_{\alpha 1},\underline{L}_{\beta 1}) \overset{7}{\le} \operatorname{Max}\{h_{\alpha}(\underline{L}_{\alpha 1}),h_{\beta}(\underline{L}_{\beta 1})\} \overset{8}{\le} H_{\Phi}$

$H_{\Phi} \overset{9}{\le} \min\{h_{\alpha}(\underline{U}_{\alpha 1}),h_{\beta}(\underline{U}_{\beta 1})\} \overset{10}{\le} u_{\alpha\beta 3}(\underline{U}_{\alpha 1},\underline{U}_{\beta 1})$.

Proof: The first part of this result can be obtained as an immediate consequence of corollary 3.8 or 4.2. In addition, the inequalities 7 to 10 are immediate: by corollary A.6.10 (see (4.1) and (4.2)), $H_\Phi = h_\alpha(\underline{H}_\alpha) = h_\beta(\underline{H}_\beta)$ and by theorem A.6.3, h_α and h_β are non-decreasing in each of their arguments; consequently,

$l_{\alpha 1}(\underline{L}_{\alpha 1}) \leq h_\alpha(\underline{L}_{\alpha 1}) \leq h_\alpha(\underline{H}_\alpha)$ and $l_{\beta 2}(\underline{L}_{\beta 1}) \leq h_\beta(\underline{L}_{\beta 1}) \leq h_\beta(\underline{H}_\beta)$;

this yields the inequalities 7 and 8; the inequalities 9 and 10 follow immediately, by duality $\square$

Some further "bounds" for the availability could be equally deduced, in the same way, from corollary 3.9; however, they would have a poor interest: with the dependence conditions then to be considered, some "tighter bounds" can be obtained by "composing the bounds of type 2". This is shown with the next theorem and its corollary. First, some lemma will appear helpful; it can be immediately obtained by restricting "lemma A.3.2" in [20] to the modular decompositions with a parallel organizing system.

Lemma 4.4: Let $\{(M_r,\mu_r) \,/\, r=1,\dots,m\}$ ($m \in \mathbf{N}^*$) be a modular decomposition for some binary coherent system, (C,Φ), with a parallel organizing system; if the performance variables of the modules, $\mu_r(\underline{X}_{M_r})$, $r=1,\dots,m$, are mutually independent and if the joint performance process of the components of each module, $\underline{X}_{M_r}$, $r=1,\dots,m$, is associated, then, $L_{\Phi 2} \leq \coprod_{r=1}^{m} L_{\mu_r 2}$.

The next result extends "lemma A.3.2" in [20] and consequently, "theorems 2 and 6" in [5].

Theorem 4.5: ("Composition of the Bounds of Type 2")

Under assumptions 3.1, if the pseudo-modular decompositions $\mathcal{A}$ and $\mathcal{B}$ satisfy all the dependence conditions stated in corollary 4.2 and if in addition,

(4.4) the performance vector of the components of each pseudo-module, $\underline{X}_{A_r}$, $r=1,\dots,a$, $\underline{X}_{B_r}$, $r=1,\dots,b$, is associated,

then, $L_{\Phi 2} \overset{1}{\leq} l_{\beta 2}(\underline{L}_{\beta 2}) \overset{2}{\leq} l_{\beta 2}(\underline{H}_\beta) \overset{3}{\leq} H_\Phi \overset{4}{\leq} u_{\alpha 2}(\underline{H}_\alpha) \overset{5}{\leq} u_{\alpha 2}(\underline{U}_{\alpha 2}) \overset{6}{\leq} U_{\Phi 2}$.

Proof: First, note the inequalities 4 to 6 can be immediately deduced from the inequalities 1 to 3, by duality. The non-decreasing property of $l_{\beta 2}$ in each of its arguments yields immediately the inequality 2. According to (3.2), it suffices to apply the "bounds" of type 2 to the organizing system (B,β) for obtaining the inequality 3. So, the main part of the proof concerns the inequality 1 which can be proved with an approach analogous to the one proposed in [5] and equally used in [20] (see Appendix); but a shorter proof can be proposed by means of lemma 4.4. Indeed, it suffices to apply it to the modular decomposition $\{(B_s,\beta_{sv_s}) \,/\, (s,v_s)\in K'_r\}$ of the binary coherent system (C,Φ_r) defined in remark 3.3, for each $r = 1,\dots,k'$: $L_{\Phi_r 2} \le \coprod_{(s,v_s)\in K'_r} L_{\beta_{sv_s} 2}$; consequently (see (3.3)),

$$L_{\Phi 2} = \prod_{r=1}^{k} H_{\theta_r} = \prod_{r=1}^{k'} L_{\Phi_r 2} \le \prod_{r=1}^{k'} \coprod_{(s,v_s)\in K'_r} L_{\beta_{sv_s} 2} = l_{\beta 2}(\underline{L}_{\beta 2}) \qquad \square$$

Corollary 4.6: Under assumptions 3.1, all the inequalities in theorem 4.5 still hold if the conditions (4.1), (4.2) and (4.4) are satisfied. Furthermore, if in addition, the pseudo-modular decompositions $\mathcal{A}$ and $\mathcal{B}$ are monotone, then,

$$l_{\beta 2}(\underline{L}_{\beta 2}) \overset{7}{\le} h_\beta(\underline{L}_{\beta 2}) \overset{8}{\le} H_\Phi \overset{9}{\le} h_\alpha(\underline{U}_{\alpha 2}) \overset{10}{\le} \bar{u}_{\alpha 2}(\underline{U}_{\alpha 2}).$$

Proof: The first part of this result constitutes an obvious particular case of theorem 4.5: note (4.1), (4.2) and (4.4) ensure (4.3). The inequalities 7 to 10 can be easily proved by means of corollary A.6.10, with some arguments quite analogous to the ones proposed for corollary 4.3 $\square$

The next result extends "theorem 3.2" in [20].

Corollary 4.7: With the same assumptions as in theorem 4.5, for every $e\in\{1,3\}$ and for every $f\in\{2,3\}$,

$$L_{\Phi 3} \overset{1}{\le} l_{\alpha\beta 3}(\underline{L}_{\alpha e},\underline{L}_{\beta f}) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_\alpha,\underline{H}_\beta) \overset{3}{\le} H_\Phi$$

$$H_\Phi \overset{4}{\le} u_{\alpha\beta 3}(\underline{H}_\alpha,\underline{H}_\beta) \overset{5}{\le} u_{\alpha\beta 3}(\underline{U}_{\alpha f},\underline{U}_{\beta e}) \overset{6}{\le} U_{\Phi 3}.$$

Furthermore, if the assumptions of corollary 4.6 are verified, these inequalities still hold and if in addition, the pseudo-modular decompositions $\mathcal{A}$ and $\mathcal{B}$ are monotone,

$$l_{\alpha\beta 3}(\underline{L}_{\alpha e},\underline{L}_{\beta f}) \overset{7}{\le} \mathrm{Max}\{h_\alpha(\underline{L}_{\alpha e}),h_\beta(\underline{L}_{\beta f})\} \overset{8}{\le} H_\Phi$$

$H_\Phi \overset{9}{\le} \min\{h_\alpha(\underline{U}_{\alpha f}), h_\beta(\underline{U}_{\beta e})\} \overset{10}{\le} u_{\alpha\beta 3}(\underline{U}_{\alpha f}, \underline{U}_{\beta e})$.

Proof: The inequality 1 can be easily deduced from the inequalities 1 in theorems 3.11 and 4.5: for any $\underline{p}\in[0,1]^{|C|}$,

(4.5) $\quad l_{\Phi 1}(\underline{p}) = l_{\alpha 1}(\underline{L}_{\alpha 1}) \le l_{\alpha 1}(\underline{L}_{\alpha 3})$ and $L_{\Phi 2} \le l_{\beta 2}(\underline{L}_{\beta 2}) \le l_{\beta 2}(\underline{L}_{\beta 3})$

since $l_{\alpha 1}$ and $l_{\beta 2}$ are non-decreasing in each of their arguments; the inequality 1 now appears immediate:

$\mathrm{Max}\{l_{\Phi 1}(\underline{p}), L_{\Phi 2}\} \le \mathrm{Max}\{l_{\alpha 1}(\underline{L}_{\alpha 1}), l_{\beta 2}(\underline{L}_{\beta 2})\} \le \mathrm{Max}\{l_{\alpha 1}(\underline{L}_{\alpha 3}), l_{\beta 2}(\underline{L}_{\beta 3})\}$.

The non-decreasing property of $l_{\alpha\beta 3}$ in each of its arguments ensures the inequality 2 while the inequality 3 has been proved with corollary 4.2. The inequalities 4 to 6 now follow immediately, by duality. In addition, the inequalities 7 to 10 can be proved with some arguments quite analogous to the ones proposed for the inequalities with the same numbers in corollary 4.3; indeed, these inequalities correspond to the particular case e = f = 1.

The next corollary follows immediately; indeed, it generalizes the "binary version" of "corollary 5.3" proposed in [15].

Corollary 4.8: Under assumptions 3.1, if the performance variables of the components of (C,Φ), X_i, $i\in C$, satisfy the same conditions of dependence as in corollary 2.15 and if in addition,

(a) the joint performance vectors of the components of the pseudo-modules in $\mathcal{A}$, $\underline{X}_{A_r}$, r = 1,...,a, are mutually independent,

(b) the joint performance vectors of the components of the pseudo-modules in $\mathcal{B}$, $\underline{X}_{B_r}$, r = 1,...,b, are mutually independent,

then, for every $e\in\{1,3\}$, every $f\in\{2,3\}$ and any $\underline{p}\in[0,1]^{|C|}$,

$l_{\Phi 3}(\underline{p}) \overset{1}{\le} l_{\alpha\beta 3}(\underline{l}_{\alpha e}(\underline{p}), \underline{l}_{\beta f}(\underline{p})) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_\alpha, \underline{H}_\beta) \overset{3}{\le} H_\Phi$

$H_\Phi \overset{4}{\le} u_{\alpha\beta 3}(\underline{H}_\alpha, \underline{H}_\beta) \overset{5}{\le} u_{\alpha\beta 3}(\underline{u}_{\alpha f}(\underline{p}), \underline{u}_{\beta e}(\underline{p})) \overset{6}{\le} u_{\Phi 3}(\underline{p})$.

If in addition, the pseudo-modular decompositions $\mathcal{A}$ and $\mathcal{B}$ are monotone, $l_{\alpha\beta 3}(\underline{l}_{\alpha e}(\underline{p}), \underline{l}_{\beta f}(\underline{p})) \overset{7}{\le} \mathrm{Max}\{h_\alpha(\underline{l}_{\alpha e}(\underline{p})), h_\beta(\underline{l}_{\beta f}(\underline{p}))\} \overset{8}{\le} H_\Phi$

$$H_\Phi \overset{9}{\le} \min\{h_\alpha(\underline{u}_{\alpha f}(\underline{p})), h_\beta(\underline{u}_{\beta e}(\underline{p}))\} \overset{10}{\le} u_{\alpha\beta 3}(\underline{u}_{\alpha f}(\underline{p}), \underline{u}_{\beta e}(\underline{p})).$$

Proof: This result is an immediate consequence of corollary 4.7 (the assumptions of which are satisfied) since the assumptions ensure the following inequalities:

(4.6) $L_{\Phi 3} = l_{\Phi 3}(\underline{p})$ and $U_{\Phi 3} = u_{\Phi 3}(\underline{p})$ and for every $e = 1,2,3$,
$\underline{L}_{\alpha e} = \underline{l}_{\alpha e}(\underline{H}_{\alpha})$, $\underline{U}_{\alpha e} = \underline{u}_{\alpha e}(\underline{H}_{\alpha})$, $\underline{L}_{\beta e} = \underline{l}_{\beta e}(\underline{H}_{\beta})$, $\underline{U}_{\beta e} = \underline{u}_{\beta e}(\underline{H}_{\beta})$.

Remark 4.9: Of course, apart from the ones numbered 1 and 6, the inequalities in corollary 4.7 or 4.8 equally hold for $(e,f) = (1,1)$. However, the corresponding lower (upper) "bounds" appear to be better than $L_{\Phi 1}$ or $l_{\Phi 1}(\underline{p})$ ($U_{\Phi 1}$ or $u_{\Phi 1}(\underline{p})$), only (see Corollary 3.9).

5. COMPARISONS BETWEEN SOME "BOUNDS" FOR THE AVAILABILITY:

Some comparative results can be obtained between the "bounds" for the availability when they are determined from two pseudo-modular decompositions of (C,Φ) with respect to $\mathcal{H}(\Phi)$ ($\mathcal{K}(\Phi)$) defined with some comparable partitions of C. This viewpoint is examined throughout this section, thus achieving a full generalization of the results proposed till now in terms of modular decompositions [5][20].

Assumptions 5.1: Throughout the following, some elements of the pseudo-modular decompositions $\mathcal{A}$ and $\mathcal{B}$ (see Assumptions 3.1) are assumed to be decomposed further as specified hereafter.
Let $I = \{r\in\mathbf{N}^* \ / \ r\le a'\}$ and $A' = U(a';(a_r)_{r\in I})$, for some $a'\in\mathbf{N}^*$, $a' \le a$; let $D = U(d;(d_r)_r)$, for some $d\in\mathbf{N}^*$ and $(d_r)_r\in\mathbf{N}^{*d}$.
Let $\{I_r \ / \ r\in I\}$ be some partition of $\{j\in\mathbf{N}^* \ / \ j\le d\}$ and let $\{(D_s,\delta_{su_s}) \ / \ (s,u_s)\in D\}$ be some collection of binary coherent systems which satisfies the following conditions: for each $(r,v_r)\in A'$,

(a) $\{D_s \ / \ s\in I_r\}$ is a partition of A_r;

(b) there exists some $I_{rv_r} \subset I_r$ and for every $s\in I_{rv_r}$ respectively, there exists some $V_{rv_rs} \subset \{j\in\mathbf{N}^* \ / \ j\le d_s\}$ such that $\{\{(D_s,\delta_{su_s}) \ / \ u_s\in V_{rv_rs}\} \ / \ s\in I_{rv_r}\}$ is a pseudo-modular decomposition of (A_r,α_{rv_r}) with respect to $\mathcal{H}(\alpha_{rv_r})$.

For every $(r,v_r)\in A'$ respectively, the corresponding organizing system $(\bigcup_{s\in I_{rv_r}} \{s\}\times V_{rv_rs}, \varrho_{rv_r})$ verifies the following relations:

$$(5.1) \quad \begin{aligned} \alpha_{rv_r}(\underline{X}_{A_r}) &\overset{as}{=} \varrho_{rv_r}((\delta_{su_s}(\underline{X}_{D_s}))_{s\in I_{rv_r},u_s\in V_{rv_rs}}) \\ H_{\alpha_{rv_r}}[\mathcal{L}(\underline{X}_{A_r})] &= H_{\varrho_{rv_r}}[\mathcal{L}((\delta_{su_s}(\underline{X}_{D_s}))_{s\in I_{rv_r},u_s\in V_{rv_rs}})]. \end{aligned}$$

In an analogous way, let $J = \{r\in\mathbf{N}^* \,/\, r\leq b'\}$, for some $b'\in\mathbf{N}^*$, $b' \leq b$; let $B' = U(b';(b_r)_{r\in J})$; let $E = U(e;(e_r)_r)$, for some $e\in\mathbf{N}^*$ and $(e_r)_r\in\mathbf{N}^{*e}$; let $\{J_r \,/\, r\in J\}$ be some partition of $\{j\in\mathbf{N}^* \,/\, j\leq e\}$; in addition, let $\{(E_s,\varepsilon_{su_s}) \,/\, (s,u_s)\in E\}$ be some collection of binary coherent systems such that for each $(r,v_r)\in B'$, $\{E_s \,/\, s\in J_r\}$ be a partition of B_r and for some $J_{rv_r}\subset J_r$ and some $W_{rv_rs}\subset\{j\in\mathbf{N}^* \,/\, j\leq e_s\}$, $s\in J_{rv_r}$ respectively, $\{\{(E_s,\varepsilon_{su_s}) \,/\, u_s\in W_{rv_rs}\} \,/\, s\in J_{rv_r}\}$ be a pseudo-modular decomposition of (B_r,β_{rv_r}) with respect to $\mathcal{K}(\beta_{rv_r})$:

$$(5.2)\quad \begin{aligned} &\beta_{rv_r}(\underline{X}_{B_r}) \overset{as}{=} \sigma_{rv_r}((\varepsilon_{su_s}(\underline{X}_{E_s}))_{s\in J_{rv_r},u_s\in W_{rv_rs}}) \\ &H_{\beta_{rv_r}}[\mathcal{L}(\underline{X}_{B_r})] = H_{\sigma_{rv_r}}[\mathcal{L}((\varepsilon_{su_s}(\underline{X}_{E_s}))_{s\in J_{rv_r},u_s\in W_{rv_rs}})]. \end{aligned}$$

Indeed, if the pseudo-modular decomposition $\mathcal{A}(\mathcal{B})$ is strict, then, for every $(r,v_r)\in A'$ ($(r,v_r)\in B'$), $I_{rv_r} = I_r$ ($J_{rv_r} = J_r$). For every $r = 1,\ldots,a'$ ($r = 1,\ldots,b'$), some elements of the pseudo-module $\{(A_r,\alpha_{rv_r}) \,/\, v_r=1,..,a_r\}$ ($\{(B_r,\beta_{rv_r}) \,/\, v_r=1,..,b_r\}$) may include some "similar parts" and therefore, $\{V_{rv_rs} \,/\, v_r=1,\ldots,a_r\}$ ($\{W_{rv_rs} \,/\, v_r=1,\ldots,b_r\}$) is not necessarily a partition (but a covering) of $\{j\in\mathbf{N}^* \,/\, j\leq d_s\}$ ($\{j\in\mathbf{N}^* \,/\, j\leq e_s\}$), $s\in I_r$ ($s\in J_r$) respectively.
$\mathcal{D} = \{\{(D_s,\delta_{sv_s})/1\leq v_s\leq d_s\} \,/\, 1\leq s\leq d\}\cup\{\{(A_r,\alpha_{rv_r})/v_r=1,..,a_r\}/r\in\bar{I}\}$ and $\mathcal{E} = \{\{(E_s,\varepsilon_{sv_s})/1\leq v_s\leq e_s\} \,/\, 1\leq s\leq e\}\cup\{\{(B_r,\beta_{rv_r})/v_r=1,..,b_r\}/r\in\bar{J}\}$ are two pseudo-modular decompositions of (C,Φ) with respect to $\mathcal{H}(\Phi)$ and $\mathcal{K}(\Phi)$, respectively (see Proposition A.3.25) and by (3.1),

$$(5.3)\quad \begin{aligned} \Phi(\underline{X}) &\overset{as}{=} \alpha((\rho_{rv_r}((\delta_{su_s}(\underline{X}_{D_s}))_{su_s}))_{(r,v_r)\in A'},(\alpha_{rv_r}(\underline{X}_{A_r}))_{(r,v_r)\in\bar{A}'}) \\ &\overset{as}{=} \gamma((\delta_{sv_s}(\underline{X}_{D_s}))_{(s,v_s)\in D},(\alpha_{rv_r}(\underline{X}_{A_r}))_{(r,v_r)\in\bar{A}'}); \end{aligned}$$

$$(5.4)\quad \begin{aligned} \Phi(\underline{X}) &\overset{as}{=} \beta((\sigma_{rv_r}((\varepsilon_{su_s}(\underline{X}_{E_s}))_{su_s}))_{(r,v_r)\in B'},(\beta_{rv_r}(\underline{X}_{B_r}))_{(r,v_r)\in\bar{B}'}) \\ &\overset{as}{=} \kappa((\varepsilon_{sv_s}(\underline{X}_{E_s}))_{(s,v_s)\in E},(\beta_{rv_r}(\underline{X}_{B_r}))_{(r,v_r)\in\bar{B}'}). \end{aligned}$$

Remark 5.2: For every $r = 1,..,a'$, let $G_r = \bigcup_{s\in I_r}\{s\}\times\{j\in\mathbf{N}^* \,/\, j\leq d_s\}$ and $\mathcal{G} = \{\{(G_r,\rho_{rv_r})/v_r=1,\ldots,a_r\}/r\in I\}\cup\{\{(\{(r,v_r)\},i)/v_r=1,..,a_r\}/r\in\bar{I}\}$, where i denotes the identity defined on S.

It can be readily checked that $\mathcal{G}$ is a pseudo-modular decomposition of $(D\cup\bar{A}',\gamma)$ with respect to $\mathcal{H}(\gamma)$ (in particular, see the proof of proposition A.3.25); (A,α) is its organizing system (see (5.3)) and by corollary A.3.18, $H\in\mathcal{H}(\gamma)$ if and only if $H = \bigcup_{(r,v_r)\in H'} H_{rv_r}$, for some $H'\in\mathcal{H}(\alpha)$ and some $H_{rv_r}\in\mathcal{H}(\rho_{rv_r})$ if $(r,v_r)\in A'$ or some $H_{rv_r}\in\mathcal{H}(\alpha_{rv_r})$ if $(r,v_r)\in\bar{A}'$.

Of course, an analogous result can be obtained from $\mathcal{B}$ and $\mathcal{E}$: for every $r = 1,.., b'$, let $G'_r = \bigcup_{s\in J_r} \{s\}\times\{j\in\mathbf{N}^* / j\le e_s\}$; then,

$\mathcal{G}' = \{\{(G'_r,\sigma_{rv_r})/v_r=1,...,b_r\}/r\in J\}\cup\{\{(\{(r,v_r)\},i)/v_r=1,..,b_r\}/r\in\bar{J}\}$,

is a pseudo-modular decomposition of $(E\cup\bar{B}',\kappa)$ with respect to $\mathcal{J}(\kappa)$ and with the organizing system (B,β) (see (5.4)).

Notations 5.3: Some conventions stated in notations A.6.2 will appear convenient: for instance, in terms of $\mathcal{A}$ and of $\mathcal{B}$,

$(\underline{H}_{\alpha A'},\underline{L}_{\alpha 1\bar{A}'}) = ((H_{\alpha_{rv_r}})_{(r,v_r)\in A'},(L_{\alpha_{rv_r}1})_{(r,v_r)\in\bar{A}'})$,

$(\underline{L}_{\delta 1D},\underline{L}_{\alpha 1\bar{A}'}) = ((L_{\delta_{rv_r}1})_{(r,v_r)\in D},(L_{\alpha_{rv_r}1})_{(r,v_r)\in\bar{A}'})$, ...

Theorem 5.4: Under assumptions 5.1, if

(a) for each minimal path set of $(D\cup\bar{A}',\gamma)$, $H\in\mathcal{H}(\gamma)$, the performance variables $\delta_{rv_r}(\underline{X}_{D_r})$ or $\alpha_{rv_r}(\underline{X}_{A_r})$, $(r,v_r)\in H$, are mutually independent,

(b) for each minimal cut set of $(E\cup\bar{B}',\kappa)$, $K\in\mathcal{K}(\kappa)$, the performance variables $\varepsilon_{rv_r}(\underline{X}_{E_r})$ or $\beta_{rv_r}(\underline{X}_{B_r})$, $(r,v_r)\in K$, are mutually independent,

(c) the joint performance vector of all the pseudo-modules in $\mathcal{D}$, $((\delta_{rv_r}(\underline{X}_{D_r}))_{(r,v_r)\in D},(\alpha_{rv_r}(\underline{X}_{A_r}))_{(r,v_r)\in\bar{A}'})$, is associated,

(d) the joint performance vector of all the pseudo-modules in $\mathcal{E}$, $((\varepsilon_{rv_r}(\underline{X}_{E_r}))_{(r,v_r)\in E},(\beta_{rv_r}(\underline{X}_{B_r}))_{(r,v_r)\in\bar{B}'})$, is associated,

then, $L_{\Phi 1} \overset{1}{\le} l_{\gamma 1}(\underline{H}_{\delta D},\underline{L}_{\alpha 1\bar{A}'}) \overset{2}{\le} l_{\alpha 1}(\underline{H}_{\alpha A'},\underline{L}_{\alpha 1\bar{A}'}) \overset{3}{\le} H_{\Phi}$

$H_{\Phi} \overset{4}{\le} u_{\beta 1}(\underline{H}_{\beta B'},\underline{U}_{\beta 1\bar{B}'}) \overset{5}{\le} u_{\kappa 1}(\underline{H}_{\varepsilon E},\underline{U}_{\beta 1\bar{B}'}) \overset{6}{\le} U_{\Phi 1}$.

Proof: This result can be easily checked by means of theorem 3.5 all the assumptions of which are verified in the time interval $\tau(I) =[t,t]$ by the pseudo-modular decompositions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ (see Remark 2.8). In particular, if $\mathcal{D}$ ($\mathcal{E}$) satisfies the condition (a) (condition (b)), then, for each minimal path (cut) set of (A,α), $H\in\mathcal{H}(\alpha)$ (of (B,β), $K\in\mathcal{K}(\beta)$), the random variables $\alpha_{rv_r}(\underline{X}_{A_r})$,

$(r,v_r)\in H$ $(\beta_{rv_r}(\underline{X}_{B_r})$, $(r,v_r)\in K)$, are mutually independent. Indeed, it suffices to check it with respect to the minimal path sets, for instance (see Proposition A.3.16). This can be done by contraposition: assume that for some $H\in \mathcal{H}(\alpha)$, $(r,v_r)\in H$ and $(s,v_s)\in(H\setminus\{(r,v_r)\})$ the random variables $\alpha_{rv_r}(\underline{X}_{A_r})$ and $\alpha_{sv_s}(\underline{X}_{A_s})$, are dependent; by (5.1), if $(r,v_r)\in A'$ and $(s,v_s)\in A'$, then, for some $H_1\in\mathcal{H}(\varrho_{rv_r})$, $H_2\in\mathcal{H}(\varrho_{sv_s})$, $(v,u_v)\in H_1$ and $(w,u_w)\in H_2$, the random variables $\delta_{vu_v}(\underline{X}_{D_v})$ and $\delta_{wu_w}(\underline{X}_{D_w})$ are dependent; but, this is inconsistent with the condition (a) imposed on $\mathscr{D}$ since by remark 5.2, $H_1 \cup H_2 \subset H_0$, for some $H_0\in\mathcal{H}(\gamma)$; if $(r,v_r)\in\bar{A}'$ or $(s,v_s)\in\bar{A}'$, the result can be shown in an analogous way.

Indeed, it suffices to prove the inequalities 1 to 3, for instance since all the other ones follow immediately, by duality. The equality 1 in theorem 3.5 applied to the pseudo-modular decomposition $\mathscr{D}$ yields the inequality 1: $L_{\Phi 1} = l_{\gamma 1}(\underline{L}_{\delta 1 D},\underline{L}_{\alpha 1\bar{A}'}) \le l_{\gamma 1}(\underline{H}_{\delta D},\underline{L}_{\alpha 1\bar{A}'})$, since $l_{\gamma 1}$ is non-decreasing in each of its arguments. The inequality 3 results immediately from the inequality 3 in theorem 3.5 applied to the pseudo-modular decomposition $\mathscr{A}$: by the non-decreasing property of $l_{\alpha 1}$, $l_{\alpha 1}(\underline{H}_{\alpha A'},\underline{L}_{\alpha 1\bar{A}'}) \le l_{\alpha 1}(\underline{H}_{\alpha}) \le H_\Phi$. So, the main part of the proof concerns the inequality 2: let $\underline{v}_{\bar{A}'} = \underline{L}_{\alpha 1\bar{A}'}$;

$$
\begin{aligned}
(5.5)\quad l_{\gamma 1}(\underline{H}_{\delta D},\underline{v}_{\bar{A}'}) &\le l_{\gamma 1}(\underline{H}_{\delta D},\underline{H}_{\alpha\bar{A}'}) && \text{(since } l_{\gamma 1} \text{ is non-decreasing)}\\
&\le L_{\gamma 1} && \text{(by definition of } l_{\gamma 1}\text{; see Corollary 2.12)}\\
&\le l_{\alpha 1}(\underline{L}_{\varrho 1 A'},\underline{v}_{\bar{A}'})
\end{aligned}
$$

(by applying the equality 1 in theorem 3.5 to the pseudo-modular decomposition $\mathscr{G}$ of $(D\cup\bar{A}',\gamma)$; see Remark 5.2)

$$
\begin{aligned}
&\le l_{\alpha 1}(\underline{H}_{\varrho A'},\underline{v}_{\bar{A}'}) && \text{(since } l_{\alpha 1} \text{ is non-decreasing)}\\
&\le l_{\alpha 1}(\underline{H}_{\alpha A'},\underline{v}_{\bar{A}'}) && \text{(by (5.1))} \qquad \square
\end{aligned}
$$

Both of the following corollaries jointly generalize "theorem 3.3" in [20].

Corollary 5.5: With the same assumptions as in theorem 5.4,

$$L_{\Phi 1} \overset{1}{\le} l_{\gamma\beta 3}(\underline{H}_{\delta D},\underline{L}_{\alpha 1\bar{A}'},\underline{L}_{\beta 1}) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha A'},\underline{L}_{\alpha 1\bar{A}'},\underline{L}_{\beta 1}) \overset{3}{\le} H_\Phi$$

$$H_\Phi \overset{4}{\le} u_{\alpha\beta 3}(\underline{U}_{\alpha 1},\underline{H}_{\beta B'},\underline{U}_{\beta 1\bar{B}'}) \overset{5}{\le} u_{\alpha\kappa 3}(\underline{U}_{\alpha 1},\underline{H}_{\varepsilon E},\underline{U}_{\beta 1\bar{B}'}) \overset{6}{\le} U_{\Phi 1}.$$

Proof: The inequalities 1 and 3 respectively follow immediately from the inequalities 1 and 3 in corollary 4.2, respectively applied to the pseudo-modular decompositions $\mathcal{D}$ and $\mathcal{A}$:

$L_{\Phi 1} \leq l_{\gamma\beta 3}(\underline{L}_{\delta 1D}, \underline{L}_{\alpha 1\bar{A}'}, \underline{L}_{\beta 1}) \leq l_{\gamma\beta 3}(\underline{H}_{\delta D}, \underline{L}_{\alpha 1\bar{A}'}, \underline{L}_{\beta 1})$

$l_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{L}_{\alpha 1\bar{A}'}, \underline{L}_{\beta 1}) \leq l_{\alpha\beta 3}(\underline{H}_{\alpha}, \underline{H}_{\beta}) \leq H_{\Phi}$.

since $l_{\gamma\beta 3}$ and $l_{\alpha\beta 3}$ are non-decreasing in each of their arguments.
The inequality 2 in theorem 5.4 immediately yields the inequality 2:

$\mathrm{Max}\{l_{\gamma 1}(\underline{H}_{\delta D}, \underline{L}_{\alpha 1\bar{A}'}), l_{\beta 2}(\underline{L}_{\beta 1})\} \leq \mathrm{Max}\{l_{\alpha 1}(\underline{H}_{\alpha A'}, \underline{L}_{\alpha 1\bar{A}'}), l_{\beta 2}(\underline{L}_{\beta 1})\}$.

The inequalities 4 to 6 follow immediately, by duality □

Corollary 5.6: Under assumptions 5.1, all the inequalities in theorem 5.4 and in corollary 5.5 still hold if the following conditions are satisfied:

(5.6) the performance vectors of all the pseudo-modules in $\mathcal{D}$,
$(\delta_{rv_r}(\underline{X}_{D_r}))_{v_r=1,..,d_r}$, $r = 1,..,d$, $(\alpha_{rv_r}(\underline{X}_{A_r}))_{v_r=1,..,a_r}$, $r \in \bar{I}$,
are mutually independent,

(5.7) the performance vectors of all the pseudo-modules in $\mathcal{E}$,
$(\varepsilon_{rv_r}(\underline{X}_{E_r}))_{v_r=1,..,e_r}$, $r = 1,..,e$, $(\beta_{rv_r}(\underline{X}_{B_r}))_{v_r=1,..,b_r}$, $r \in \bar{J}$,
are mutually independent,

(5.8) each of these performance vectors is associated.

Furthermore, if in addition, the pseudo-modular decomposition $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are monotone, then,

$$h_{\alpha}(\underline{L}_{\alpha 1}) \overset{7}{\leq} h_{\alpha}(\underline{L}_{\alpha 1A'}, \underline{H}_{\alpha\bar{A}'}) \overset{8}{\leq} h_{\gamma}(\underline{L}_{\delta 1D}, \underline{H}_{\alpha\bar{A}'}) \overset{9}{\leq} H_{\Phi}$$

$$H_{\Phi} \overset{10}{\leq} h_{\kappa}(\underline{U}_{\varepsilon 1E}, \underline{H}_{\beta\bar{B}'}) \overset{11}{\leq} h_{\beta}(\underline{U}_{\beta 1B'}, \underline{H}_{\beta\bar{B}'}) \overset{12}{\leq} h_{\beta}(\underline{U}_{\beta 1}).$$

Proof: The first part of this result is an obvious consequence of theorem 5.4 and of corollary 5.5. Furthermore, the inequality 7 results immediately from the non-decreasing property of the performance function h_{α} (see Corollary A.6.10). The inequality 8 in corollary 4.3 applied to $\mathcal{D}$ immediately yields the inequality 9: $h_{\gamma}(\underline{L}_{\delta 1D}, \underline{H}_{\alpha\bar{A}'}) \leq h_{\gamma}(\underline{H}_{\delta D}, \underline{H}_{\alpha\bar{A}'}) \leq H_{\Phi}$, since the performance function h_{γ} is non-decreasing in each of its arguments. The inequality 8 can be easily checked as follows: as it has been shown in the course of the proof proposed for the inequality 8 in corollary 4.3, for every $(r,v_r) \in A'$, $L_{\alpha_{rv_r}1} \leq h_{\rho_{rv_r}}((\underline{L}_{\delta_{su_s}1})_{s \in I_{rv_r}, u_s \in V_{rv_rs}})$; consequently, by the non-decreasing property of the performance function h_{α},

$$h_\alpha(\underline{L}_{\alpha 1 A'}, \underline{H}_{\alpha \bar{A}'}) \leq h_\alpha(\underline{h}_{\varrho_{r v_r}}(\underline{L}_{\delta_{s u_s} 1})_{s \in I_{r v_r}, u_s \in V_{r v_r s}}, \underline{H}_{\alpha \bar{A}'})$$
$$\leq h_\gamma(\underline{L}_{\delta 1 D}, \underline{H}_{\alpha \bar{A}'}) \qquad \text{(see (5.3))}.$$

The inequalities 10 to 12 follow immediately, by duality □

The next theorem partly extends "theorems 3 and 7" in [5].

Theorem 5.7: With the same assumptions as in theorem 5.4, if in addition,

(5.9) for each pseudo-module in $\mathcal{A}$ or in $\mathcal{B}$, the performance vector of its components, $\underline{X}_{A_r}$, $r = 1,\dots,a$, $\underline{X}_{B_r}$, $r = 1,\dots,b$, is associated,

then, $L_{\Phi 2} \overset{1}{\leq} l_{\kappa 2}(\underline{H}_{\varepsilon E}, \underline{L}_{\beta 2 \bar{B}'}) \overset{2}{\leq} l_{\beta 2}(\underline{H}_{\beta B'}, \underline{L}_{\beta 2 \bar{B}'}) \overset{3}{\leq} H_\Phi$

$H_\Phi \overset{4}{\leq} u_{\alpha 2}(\underline{H}_{\alpha A'}, \underline{U}_{\alpha 2 \bar{A}'}) \overset{5}{\leq} u_{\gamma 2}(\underline{H}_{\delta D}, \underline{U}_{\alpha 2 \bar{A}'}) \overset{6}{\leq} U_{\Phi 2}$.

Proof: This result can be checked by means of theorem 4.5 with some arguments quite analogous to the ones proposed for theorem 5.4. First, note the pseudo-modular decompositions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{E}$ as also $\mathcal{G}'$ (see Remark 5.2) satisfy the conditions of dependence then required. The inequalities 1 and 3 result immediately from the inequalities 1 and 3 in theorem 4.5, respectively applied to $\mathcal{E}$ and $\mathcal{B}$: $L_{\Phi 2} \leq l_{\kappa 2}(\underline{L}_{\varepsilon 2 E}, \underline{L}_{\beta 2 \bar{B}'}) \leq l_{\kappa 2}(\underline{H}_{\varepsilon E}, \underline{L}_{\beta 2 \bar{B}'})$ and $l_{\beta 2}(\underline{H}_{\beta B'}, \underline{L}_{\beta 2 \bar{B}'}) \leq l_{\beta 2}(\underline{H}_\beta) \leq H_\Phi$, since $l_{\kappa 2}$ and $l_{\beta 2}$ are non-decreasing in each of their arguments. So, the main part of the proof concerns the inequality 2: let $\underline{w}_{\bar{B}'} = \underline{L}_{\beta 2 \bar{B}'}$;

(5.10) $l_{\kappa 2}(\underline{H}_{\varepsilon E}, \underline{w}_{\bar{B}'}) \leq l_{\kappa 2}(\underline{H}_{\varepsilon E}, \underline{H}_{\beta \bar{B}'})$ (since $l_{\kappa 2}$ is non-decreasing)

$\leq L_{\kappa 2}$ (by definition of $l_{\kappa 2}$; see Corollary 2.15)

$\leq l_{\beta 2}(\underline{L}_{\sigma 2 B'}, \underline{w}_{\bar{B}'})$

(by applying the inequality 1 in theorem 4.5 to the pseudo-modular decomposition $\mathcal{G}'$ of $(E \cup \bar{B}', \kappa)$; see Remark 5.2)

$\leq l_{\beta 2}(\underline{H}_{\sigma B'}, \underline{w}_{\bar{B}'})$ (since $l_{\beta 2}$ is non-decreasing)

$\leq l_{\beta 2}(\underline{H}_{\beta B'}, \underline{w}_{\bar{B}'})$ (by (5.2));

Of course, the inequalities 4 to 6 follow immediately, by duality □

With the same assumptions as in theorem 5.7 and by theorems 5.4 and 5.7, the following inequalities hold:

$$L_{\Phi 3} \leq l_{\gamma\kappa 3}(\underline{H}_{\delta D}, \underline{L}_{\alpha 1 \bar{A}'}, \underline{H}_{\varepsilon E}, \underline{L}_{\beta 2 \bar{B}'}) \leq l_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{L}_{\alpha 1 \bar{A}'}, \underline{H}_{\beta B'}, \underline{L}_{\beta 2 \bar{B}'}) \leq H_\Phi$$
$$H_\Phi \leq u_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{U}_{\alpha 2 \bar{A}'}, \underline{H}_{\beta B'}, \underline{U}_{\beta 1 \bar{B}'}) \leq u_{\gamma\kappa 3}(\underline{H}_{\delta D}, \underline{U}_{\alpha 2 \bar{A}'}, \underline{H}_{\varepsilon E}, \underline{U}_{\beta 1 \bar{B}'}) \leq U_{\Phi 3}.$$

But a more general result holds and extends "theorem 3.4" in [20].

Theorem 5.8: With the same assumptions as in theorem 5.7, for every f = 1 or 3 and for every g = 2 or 3,

$$L_{\Phi 3} \overset{1}{\le} l_{\gamma\kappa 3}(\underline{H}_{\delta D}, \underline{L}_{\alpha f \bar{A}'}, \underline{H}_{\varepsilon E}, \underline{L}_{\beta g \bar{B}'}) \overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{L}_{\alpha f \bar{A}'}, \underline{H}_{\beta B'}, \underline{L}_{\beta g \bar{B}'}) \overset{3}{\le} H_{\Phi}$$

$$H_{\Phi} \overset{4}{\le} u_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{U}_{\alpha g \bar{A}'}, \underline{H}_{\beta B'}, \underline{U}_{\beta f \bar{B}'}) \overset{5}{\le} u_{\gamma\kappa 3}(\underline{H}_{\delta D}, \underline{U}_{\alpha g \bar{A}'}, \underline{H}_{\varepsilon E}, \underline{U}_{\beta f \bar{B}'}) \overset{6}{\le} U_{\Phi 3}.$$

Furthermore, these inequalities still hold if the pseudo-modular decompositions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ satisfy the dependence conditions (5.6), (5.7) and (5.9) and if in addition, all of them are monotone, then, for every f = 1 or 3 and for every g = 2 or 3,

$$h_{\alpha}(\underline{L}_{\alpha f}) \overset{7}{\le} h_{\alpha}(\underline{L}_{\alpha f A'}, \underline{H}_{\alpha \bar{A}'}) \overset{8}{\le} h_{\gamma}(\underline{L}_{\delta f D}, \underline{H}_{\alpha \bar{A}'}) \overset{9}{\le} H_{\Phi}$$

$$h_{\beta}(\underline{L}_{\beta g}) \overset{10}{\le} h_{\beta}(\underline{L}_{\beta g B'}, \underline{H}_{\beta \bar{B}'}) \overset{11}{\le} h_{\kappa}(\underline{L}_{\varepsilon g E}, \underline{H}_{\beta \bar{B}'}) \overset{12}{\le} H_{\Phi}$$

$$H_{\Phi} \overset{13}{\le} h_{\alpha}(\underline{U}_{\alpha g A'}, \underline{H}_{\alpha \bar{A}'}) \overset{14}{\le} h_{\gamma}(\underline{U}_{\delta g D}, \underline{H}_{\alpha \bar{A}'}) \overset{15}{\le} h_{\alpha}(\underline{U}_{\alpha g})$$

$$H_{\Phi} \overset{16}{\le} h_{\beta}(\underline{U}_{\beta f B'}, \underline{H}_{\beta \bar{B}'}) \overset{17}{\le} h_{\kappa}(\underline{U}_{\varepsilon f E}, \underline{H}_{\beta \bar{B}'}) \overset{18}{\le} h_{\beta}(\underline{U}_{\beta f})$$

Proof: The first part of this result can be proved by means of corollary 4.7 with some arguments quite analogous to the ones proposed for theorems 5.4 and 5.7. As an indication, the inequalities 1 and 3, respectively, can be easily deduced from the inequalities 1 and 3 in corollary 4.7 while the inequality 2 results from the following inequalities: $l_{\gamma 1}(\underline{H}_{\delta D}, \underline{L}_{\alpha f \bar{A}'}) \le l_{\alpha 1}(\underline{H}_{\alpha A'}, \underline{L}_{\alpha f \bar{A}'})$ and $l_{\kappa 2}(\underline{H}_{\varepsilon E}, \underline{L}_{\beta g \bar{B}'}) \le l_{\beta 2}(\underline{H}_{\beta B'}, \underline{L}_{\beta g \bar{B}'})$. Indeed, these inequalities can be deduced from (4.5) (which ensures the inequality 1 in corollary 4.7) by setting $\underline{v}_{\bar{A}'} = \underline{L}_{\alpha f \bar{A}'}$, f = 1 or 3, and $\underline{w}_{\bar{B}'} = \underline{L}_{\beta g \bar{B}'}$, g = 2 or 3, throughout the inequalities (5.5) and (5.10), respectively.
Furthermore, the inequalities 7 to 12 can be checked with an approach quite analogous to the one proposed for the inequalities 7 to 9 in corollary 5.6 and all the other inequalities follow immediately, by duality $\square$

The next corollary follows immediately (see (4.6)) and extends the "binary version of corollary 5.6 in [15].

Corollary 5.9: Under assumptions 5.1, if the performance variables of all the components of (C,Φ), X_i, $i \in C$, satisfy the same conditions of dependence as in corollary 2.15 and if in addition,

(a) the joint performance vectors of the components of the pseudo-modules in $\mathcal{D}$, $\underline{X}_{D_r}$, $r = 1,\dots,d$, are mutually independent,

(b) the joint performance vectors of the components of the pseudo-modules in $\mathcal{E}$, $\underline{X}_{E_r}$, $r = 1,\dots,e$, are mutually independent,

then, for every $f = 1$ or 3, every $g = 2$ or 3 and any $\underline{p} \in [0,1]^{|C|}$,

$$l_{\Phi 3}(\underline{p}) \overset{1}{\le} l_{\gamma\kappa 3}(\underline{H}_{\delta D}, \underline{l}_{\alpha f}(\underline{p})_{\bar{A}'}, \underline{H}_{\varepsilon E}, \underline{l}_{\beta g}(\underline{p})_{\bar{B}'})$$
$$\overset{2}{\le} l_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{l}_{\alpha f}(\underline{p})_{\bar{A}'}, \underline{H}_{\beta B'}, \underline{l}_{\beta g}(\underline{p})_{\bar{B}'}) \overset{3}{\le} H_{\Phi}$$

$$H_{\Phi} \overset{4}{\le} u_{\alpha\beta 3}(\underline{H}_{\alpha A'}, \underline{u}_{\alpha g}(\underline{p})_{\bar{A}'}, \underline{H}_{\beta B'}, \underline{u}_{\beta f}(\underline{p})_{\bar{B}'})$$
$$\overset{5}{\le} u_{\gamma\kappa 3}(\underline{H}_{\delta D}, \underline{u}_{\alpha g}(\underline{p})_{\bar{A}'}, \underline{H}_{\varepsilon E}, \underline{u}_{\beta f}(\underline{p})_{\bar{B}'}) \overset{6}{\le} u_{\Phi 3}(\underline{p}).$$

Furthermore, if the pseudo-modular decompositions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are monotone, then, for every $f = 1$ or 3, every $g = 2$ or 3 and any $\underline{p} \in [0,1]^{|C|}$,

$$h_{\alpha}(\underline{l}_{\alpha f}(\underline{p})) \overset{7}{\le} h_{\alpha}(\underline{l}_{\alpha f}(\underline{p})_{A'}, \underline{H}_{\alpha\bar{A}'}) \overset{8}{\le} h_{\gamma}(\underline{l}_{\delta f}(\underline{p})_{D}, \underline{H}_{\alpha\bar{A}'}) \overset{9}{\le} H_{\Phi}$$
$$h_{\beta}(\underline{l}_{\beta g}(\underline{p})) \overset{10}{\le} h_{\beta}(\underline{l}_{\beta g}(\underline{p})_{B'}, \underline{H}_{\beta\bar{B}'}) \overset{11}{\le} h\ (\underline{l}_{\varepsilon g}(\underline{p})_{E}, \underline{H}_{\beta\bar{B}'}) \overset{12}{\le} H_{\Phi}$$
$$H_{\Phi} \overset{13}{\le} h_{\alpha}(\underline{u}_{\alpha g}(\underline{p})_{A'}, \underline{H}_{\alpha\bar{A}'}) \overset{14}{\le} h_{\gamma}(\underline{u}_{\delta g}(\underline{p})_{D}, \underline{H}_{\alpha\bar{A}'}) \overset{15}{\le} h_{\alpha}(\underline{u}_{\alpha g}(\underline{p}))$$
$$H_{\Phi} \overset{16}{\le} h_{\beta}(\underline{u}_{\beta f}(\underline{p})_{B'}, \underline{H}_{\beta\bar{B}'}) \overset{17}{\le} h_{\kappa}(\underline{u}_{\varepsilon f}(\underline{p})_{E}, \underline{H}_{\beta\bar{B}'}) \overset{18}{\le} h_{\beta}(\underline{u}_{\beta f}(\underline{p}))$$

According to the results proposed in this section, the conclusions stated in [20] for the modular decompositions can be extended in terms of pseudo-modular decompositions: refining a pseudo-modular decomposition does not yield any improvement of the "refined bounds" which can be obtained in the general case; but, the opposite holds for the "refined bounds" which can be obtained with the monotonicity property, only (see theorem 5.8 and corollary 5.6 and 5.9).

6. SOME CONCLUSIONS:

All the "refined bounds" proposed in [20] for the interval reliability and for the availability have been generalized in terms of pseudo-modular decompositions. The fundamental results then obtained have been emphasized as theorems while their pragmatical consequences have been stated in various corollaries. In addition, it should be noted that the three lemmas proposed in terms of modular decompositions in [20] play a key part for proving the refinements yielded by the pseudo-modular decompositions.

According to corollary A.3.13, the pseudo-modular decompositions are the most general coherent decompositions which can yield some improvements for the "basic bounds" obtained till now with the minimal path sets or cut sets; but, contrary to the ones proposed in [20], most of the "refined bounds" obtained in terms of pseudo-modular decompositions concern any binary coherent system (see Remark A.4.5); the only restriction concerns the "bounds" obtained with the monotone pseudo-modular decompositions. This is quite motivating for improving the "refined bounds" obtained till now as also for extending their application domain by investigating some dependences weaker than association.

In addition, all the "basic bounds" and all the "refined bounds" for the binary case can be extended with some simple arguments to the multinary coherent systems: this is shown in the second part of this study [19], thus extending all the "refined bounds" currently proposed for the multinary case, with an approach easier than the one followed in [6] and [15] (see Preamble).

* *
*

Appendix:

A "DIRECT PROOF" FOR THE INEQUALITY 1 IN THEOREM B.4.5

Of course, all the conventions stated for theorem B.4.5 are used throughout the following.

For every $(s,v_s)\in B$, let $\mathcal{K}(\beta_{sv_s}) = \{K_{sv_su} \,/\, u=1,\ldots,k(s,v_s)\}$ and let $(K_{sv_su},\sigma_{sv_su})$ be the parallel-system corresponding to the minimal cut set K_{sv_su} of (B_s,β_{sv_s}), $u = 1,\ldots,k(s,v_s)$ respectively.

Let (D,Φ') be the binary strict-sense coherent system defined as follows: $D = \bigcup_{(s,v_s)\in B} \{(s,v_s)\}\times\{u\in N^* \,/\, u\leq k(s,v_s)\}$ and for every $\underline{y}\in S^{|D|}$ such that $\underline{y} = (\sigma_{sv_su}(\underline{x}_{B_s}))_{(s,v_s,u)\in D}$ for some $\underline{x}\in S_C$,

$$\Phi'(\underline{y}) = \prod_{r=1}^{k'} \coprod_{(s,v_s)\in K'_r} \prod_{u=1}^{k(s,v_s)} y_{sv_su} = \beta((\beta_{sv_s}(\underline{x}_{B_s}))_{(s,v_s)\in B}) = \Phi(\underline{x}).$$

In addition, the marginal performance processes $\{Y_{sv_su}(t) \,/\, t\in\tau\}$, $(s,v_s,u)\in D$, are assumed to be mutually independent and at the concerned point in time, $P[Y_{sv_su}=1] = H_{\sigma_{sv_su}}$, the availability of $(K_{sv_s},\sigma_{sv_su})$.

Therefore, the availability of (D,Φ') can be expressed as follows:

$$H_{\Phi'} = h_{\Phi'}(\underline{H}_\sigma) = \prod_{r=1}^{k'} \coprod_{(s,v_s)\in K'_r} \prod_{u=1}^{k(s,v_s)} H_{\sigma_{sv_su}} = \prod_{r=1}^{k'} \coprod_{(s,v_s)\in K'_r} L_{\beta_{sv_s}2};$$ so,

$$(1)\quad h_{\Phi'}(\underline{H}_\sigma) = 1_{\beta 2}(\underline{L}_{\beta 2}).$$

Furthermore, by definition of (D,Φ'), $K''\in\mathcal{K}(\Phi')$ if and only if: $K'' = \{(s,v_s,u(s,v_s)) \,/\, (s,v_s)\in K'\}$, for some $K'\in\mathcal{K}(\beta)$ and some

$u(s,v_s) = 1,\ldots,k(s,v_s)$, $(s,v_s) \in K'$ respectively.

So, by corollary A.3.18, there is a one-to-one correspondence between $\mathcal{K}(\Phi)$ and $\mathcal{K}(\Phi')$:

$$\mathcal{K}(\Phi) = \{ K_r = \bigcup_{(s,v_s,u) \in K''_r} K_{sv_su} \ / \ K''_r \in \mathcal{K}(\Phi'); r=1,\ldots,k\};$$

and for every $r = 1,\ldots,k$ and for every $\underline{x} \in S_C$,

$$\theta_r(\underline{x}) = \coprod_{(s,v_s,u) \in K''_r} \sigma_{sv_su}(\underline{x});$$

so, the pseudo-modularity of $\mathcal{B}$ with respect to $\mathcal{K}(\Phi)$ ensures that $\{(K_{sv_su}, \sigma_{sv_su}) \ / \ (s,v_s,u) \in K''_r\}$ is a modular decomposition of the parallel system (K_r,θ_r) with a parallel organizing system, (K''_r, ζ_r); in addition, according to the assumptions for $\mathcal{B}$, the performance variables of these modules, $\sigma_{sv_su}(\underline{X}_{B_s})$, $(s,v_s,u) \in K''_r$, are mutually independent; consequently,

$$H_{\theta_r} = P[\coprod_{(s,v_s,u) \in K''_r} \sigma_{sv_su}(\underline{X}_{B_s}) = 1] = \coprod_{(s,v_s,u) \in K''_r} H_{\sigma_{sv_su}} = h_{\zeta_r}(\underline{H}_\sigma).$$

The inequality 1 now follows immediately:

$$L_{\Phi 2} = \prod_{r=1}^{k} H_{\theta_r} = \prod_{r=1}^{k} h_{\zeta_r}(\underline{H}_\sigma) = L_{\Phi',2} \le h_{\Phi'}(\underline{H}_\sigma) = 1_{\beta 2}(\underline{L}_{\beta 2}) \text{ (see (1))} \quad \square$$

***

REFERENCES

[1] **BARLOW R.E. and PROSCHAN F.** (1965) "Mathematical Theory of Reliability", John Wiley and Sons, New York.

[2] **BARLOW R.E. and PROSCHAN F.** (1975) "Statistical Theory of Reliability and Life Testing", Vol.1: "Probability Models", Holt Rinehart and Winston, New York. Reprinted (1981) by To Begin With, Silver Springs, MD.

[3] **BIRNBAUM Z.W. and ESARY J.D.** (1965) "Modules of Coherent Binary Systems", SIAM Journal on Applied Mathematics, Vol.13, N.2, pp.444-462.

[4] **BLOCK H.W. and SAVITS T.H.** (1982) "A Decomposition for Multi-state Monotone Systems", Journal of Applied Probability, Vol.19, pp.391-402.

[5] **BODIN L.D.** (1970) "Approximations to System Reliability Using a Modular decomposition", Technometrics, Vol.12, N.2, pp.335-444.

[6] **BUTLER D.A.** (1982) "Bounding the Reliability of Multistate Systems", Journal of Operations Research, Vol.30, N.3, pp.530-544.

[7] **BUTTERWORTH R.W.** (1972) "A Set Theoretic Treatment of Coherent Systems", SIAM Journal on Applied Mathematics, Vol.22, N.4, pp.590-598.

[8] **CHATTERJEE P.** (1974) "Fault Tree Analysis: Reliability Theory and Systems Safety Analysis", ORC74-34, Operations Research Center, University of California, Berkeley.

[9] **CHATTERJEE P.** (1975) "Modularization of Fault Trees: A Method for Reducing the Cost of Reliability Analysis" in "Reliability and Fault Tree Analysis", R.E. BARLOW and N. SINGPURWALLA Editors, SIAM Conference Volume, Philadelphia, pp.101-126.

[10] **ESARY J.D. and MARSCHALL A.W.** (1964) "System Structure and the Existence of a System Life", Technometrics, Vol.6, N.4, pp.456-462.

[11] **ESARY J.D. and MARSCHALL A.W.** (1970) "Coherent Life Functions", SIAM Journal on Applied Mathematics, Vol.18, N.4, pp.810-814.

[12] **ESARY J.D. and PROSCHAN F.** (1963) "Coherent Structures of Non-Identical Components", Technometrics, Vol.5, N.2, pp.191-209.

[13] **ESARY J.D. and PROSCHAN F.** (1970) "A Reliability Bound for Systems of Maintained Interdependent Components", Journal of the American Association, Vol.65, N.329, pp.329-338.

[14] **ESARY J.D. and PROSCHAN F. and WALKUP D.W.** (1967) "Association of Random Variables with Applications", The Annals of Mathematical Statistics, Vol. 38, N.2, pp.1466-1474.

[15] **FUNNEMARK E. and NATVIG B.** (1985) "Bounds for the Availabilities in a Fixed Time Interval for Multistate Monotone Systems", Advances in Applied Probability, Vol.17, pp.638-665.

[16] **HJORT N.L., NATVIG B. and FUNNEMARK E.** (1985) "The Association in Time of a Markov Process with Application to Multistate Reliability Theory", Journal of Applied Probability, Vol.22, pp.473-479.

[17] **HUSEBY A.** (1986) "A Representation of Multivariate Binary Variables With Application to Reliability Theory", Report of the Center for Industrial Research, Oslo (Norway).

[18] **MAZARS N.** (1986a) " Multinary Systems and Reliability Models, From Coherence to Some Kind of Non-Coherence", EUR Report 10629 EN, Commission of the European Communities, Joint Research Center, ISPRA Establishment (Italy).

[19] **MAZARS N.** (1986b) "Some Efficient Deterministic Weapons Against Complexity in Reliability Theory: Coherent Subsystems and Pseudo-Modules for Coherent Systems -- Part II: the Multinary Case", Research Report (In Preparation).

[20] **NATVIG B.** (1980) "Improved Bounds for the Availability and Unavailability in a Fixed Time Interval for Systems of Maintained Interdependent Components", Advances in Applied Probability, Vol.12, pp.200-221.

[21] **ROSENTHAL A.** (1975) "A Computer Scientist Look at Reliability Computations", in "Reliability and Fault Tree Analysis", R.E. BARLOW and N. SINGPURWALLA Editors, SIAM Conference Volume, Philadelphia, pp.133-152.

[22] **RUSCHENDORF L.** "Weak Association of Random Variables", Journal of Multivariate Analysis, Vol. 11, pp.458-451.

* *
*

ERRATUM for the Statistical Research Report: "SOME EFFICIENT DETERMINISTIC WEAPONS AGAINST COMPLEXITY IN RELIABILITY THEORY: COHERENT SUBSYSTEMS AND PSEUDO-MODULES FOR COHERENT SYSTEMS; Part I: The Binary Case", by N. MAZARS.

The author must apologize for the following errors:

Preamble, p.V, lines 11-12: "yields" instead of "allows to establish".

Chapter A,

p.18, remark 3.22, lines 4 and 7-8: "with respect to both $\mathcal{H}(\Phi)$ and $\mathcal{K}(\Phi)$" instead of "both with respect to $\mathcal{H}(\Phi)$ and $\mathcal{K}(\Phi)$".

p.36, theorem 6.3, line 8: "if $i_r < i_s$, then, $X_r \overset{as}{\leq} X_s$" instead of "if $i_r \leq i_s$, then, $X_{i_r} \overset{as}{\leq} X_{i_s}$";

p.36-37, throughout theorem 6.3 and its proof: "Δ" instead of "$[0,1]^{|C|}$", with $\Delta = \{\underline{p}\in[0,1]^{|C|} / \forall j\in\{u\in N^*/u\leq c\}, \forall(r,s)\in C_j^2 : i_r < i_s, p_r \leq p_s\}$.

p.36, theorem 6.3, line 15: "for every $\underline{p}\in\Delta$, $h_{\Phi}D(\underline{1}-\underline{p}) = 1 - h_{\Phi}(\underline{p})$" instead of "for every $\underline{p}\in[0,1]^{|C|}$, $h_{\Phi}(\underline{p}) = 1 - h_{\Phi}(\underline{1}-\underline{p})$".

p.37, proof of theorem 6.3: consider the vector $(\underline{0}_{A(j,i)}, \underline{1}_{B(j,i)})$ instead of the vector $\underline{0}_{C_j\setminus\{i\}}$, with $A(j,i) = \{r\in C_j / \ell_r < \ell_i\}$ and $B(j,i) = C_j \setminus (A(j,i)\cup\{i\})$;

p.2, line 9; p.3, line 9; p.4, line 11; p.10, lines 18 and 21; p. 15, line 15; p.29, line 1; p.35, notations 6.2, line 7; p.39, line 19: "allow(s) us to" instead of "allow(s) to".

Chapter B,

p.42, line 4; p.49, line 16: "allow(s) us to" instead of "allow(s) to";

p.45, definition 2.9, line 3: "**mutually independent** in the time interval $\tau(I)$ if and only if" instead of "**mutually independent** if and only if";

.../...

p.51, theorem 3.5, (c) and (d): "the joint performance process of all the pseudo-modules" instead of "the joint performance processes of all the pseudo-modules";

p.59, corollary 4.7, lines 1-2: "for every $\underline{v}\in\{\underline{l}_{\alpha 1}(\underline{p}),\underline{L}_{\alpha 3}\}$, every $\underline{w}\in\{\underline{u}_{\beta 1}(\underline{p}),\underline{U}_{\beta 3}\}$ and every $f\in\{2,3\}$" instead of "for every $e\in\{1,3\}$ and for every $f\in\{2,3\}$,"

p.59-60, corollary 4.7, throughout the inequalities 1 to 3 and 7 to 8: "$\underline{v}$" instead of "$\underline{L}_{\alpha e}$"; **throughout the inequalities 4 to 6 and 9 to 10:** "$\underline{w}$" instead of "$\underline{L}_{\beta e}$";

p.60, proof of corollary 4.7, lines 3 and 6: $\underline{l}_{\alpha 1}(\underline{p})$ instead of $\underline{L}_{\alpha 1}$;

p.63, line 7: read "is a pseudo-modular-decomposition of $(E\cup\bar{B}',\kappa)$ with respect to $\mathcal{K}(\kappa)$";

p.65, corollary 5.6, line 11: read "Furthermore, if in addition, the pseudo-modular decompositions $\mathcal{A}$,";

p.65, proof for corollary 5.6, line 10:

$L_{\alpha_{rv_r}1} \le h_{\varrho_{rv_r}}((L_{\delta_{su_s}1})_{s\in I_{rv_r},u_s\in V_{rv_rs}})$ instead of

$L_{\alpha_{rv_r}1} \le h_{\varrho_{rv_r}}((\underline{L}_{\delta_{su_s}1})_{s\in I_{rv_r},u_s\in V_{rv_rs}})$;

p.66, line 1: read

$$h_\alpha(\underline{L}_{\alpha 1A'},\underline{H}_{\alpha\bar{A}'}) \le h_\alpha((\underline{h}_{\varrho_{rv_r}}((L_{\delta_{su_s}1})_{s\in I_{rv_r},u_s\in V_{rv_rs}}))_{(r,v_r)\in\bar{A}'},\underline{H}_{\alpha\bar{A}'})$$

p.67, theorem 5.8: instead of the inequalities 13 to 18, read:

$$H_\Phi \overset{13}{\le} h_\gamma(\underline{U}_{\delta gD},\underline{H}_{\alpha\bar{A}'}) \overset{14}{\le} h_\alpha(\underline{U}_{\alpha gA'},\underline{H}_{\alpha\bar{A}'}) \overset{15}{\le} h_\alpha(\underline{U}_{\alpha g})$$

$$H_\Phi \overset{16}{\le} h_\kappa(\underline{U}_{\varepsilon fE},\underline{H}_{\beta\bar{B}'}) \overset{17}{\le} h_\beta(\underline{U}_{\beta fB'},\underline{H}_{\beta\bar{B}'}) \overset{18}{\le} h_\beta(\underline{U}_{\beta f})$$

p.68, corollary 5.9: instead of the inequalities 13 to 18, read

$$H_\Phi \overset{13}{\le} h_\gamma(\underline{u}_{\delta g}(\underline{p})_D,\underline{H}_{\alpha\bar{A}'}) \overset{14}{\le} h_\alpha(\underline{u}_{\alpha g}(\underline{p})_{A'},\underline{H}_{\alpha\bar{A}'}) \overset{15}{\le} h_\alpha(\underline{u}_{\alpha g}(\underline{p}))$$

$$H_\Phi \overset{16}{\le} h_\kappa(\underline{u}_{\varepsilon f}(\underline{p})_E,\underline{H}_{\beta\bar{B}'}) \overset{17}{\le} h_\beta(\underline{u}_{\beta f}(\underline{p})_{B'},\underline{H}_{\beta\bar{B}'}) \overset{18}{\le} h_\beta(\underline{u}_{\beta f}(\underline{p}))$$

Oslo, December 1986